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présentée par

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# Ergodic properties of low dimensional flows including dispersive billiards

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sous la direction de **Viviane Baladi**

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**Propriétés ergodiques des flots en basses dimensions incluant les billards dispersifs**

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**Ergodic properties of low dimensional flows including dispersive billiards**

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# Propriétés ergodiques des flots en basses dimensions incluant les billards dispersifs


## Résumé

Cette thèse est divisée en deux parties. Dans la première partie, nous proposons une preuve courte montrant que la croissance des intégrales ergodiques d'un flot uniquement ergodique sur un tore en dimension deux – et admettant une section transverse dont l'application de Poincaré a un nombre de rotation de type constant – est au plus logarithmique. En appliquant ce résultat au développement asymptotique des intégrales ergodiques pour les flots de Giulietti–Liverani, nous obtenons une nouvelle preuve de l'absence de résonance de Ruelle non triviale de module strictement supérieur à un. Nous donnons également un exemple de flot sur le tore renormalisé par un difféomorphisme Axiome A, satisfaisant les hypothèses impliquant une croissance au plus logarithmique.

Dans la deuxième partie, nous construisons des états d'équilibre pour l'application de collision d'un billard dispersif, associés à des potentiels Hölder par morceaux. Cette construction repose sur l'étude d'un opérateur de transfert pondéré agissant sur des espaces de Banach anisotropes de distributions. Nous montrons que lorsque le potentiel satisfait certaines conditions techniques, alors il existe un état d'équilibre, qui de plus est unique, Bernoulli, adapté et a un support total. Nous montrons qu'il existe un potentiel particulier tel que l'ensemble de ses états d'équilibre est en bijection avec l'ensemble des mesures d'entropie maximale du flot billard. Dans la dernière partie, nous montrons que ce potentiel satisfait les hypothèses suffisantes garantissant l'existence et les autres résultats énoncés sur l'unique mesure d'équilibre. Par conséquent, nous obtenons une condition suffisante pour que le flot billard admette une unique mesure d'entropie maximale, et nous donnons des exemples de billards qui satisfont cette condition. Enfin, nous prouvons que cette mesure est Bernoulli, adaptée au flot et a un support total.



**Mots-clés :** Systèmes Dynamiques, billard dispersif, formalisme thermodynamique, opérateur de transfert, Banach anisotrope, theorie spectrale, résonance de Ruelle

# Ergodic properties of low dimensional flows including dispersive billiards


## Abstract

This thesis is divided into two parts. In the first part, we give a short proof showing that the growth of ergodic integrals of a uniquely ergodic flow on a torus in dimension two – and admitting a transverse section whose first return Poincaré map has a rotation number of constant type – is at most logarithmic. By applying this result to the asymptotic expansion of the ergodic integrals for Giulietti–Liverani flows, we obtain a new proof of the absence of non-trivial Ruelle resonance of modulus strictly larger than one. We also give an example of a flow on the torus renormalized by an Axiom A diffeomorphism, satisfying the hypotheses implying at most logarithmic growth.

In the second part, we construct equilibrium states for the collision map of a dispersive billiard, associated to piecewise Hölder potentials. This construction is based on the study of a weighted transfer operator acting on an anisotropic Banach space of distributions. We show that when the potential satisfies certain technical conditions, then the equilibrium state exists, is unique, Bernoulli, adapted and has full support. We show that there exists a potential such that the set of its equilibrium states are in bijection with the set of measures of maximal entropy of the billiard flow. In the last part, we show that this potential satisfies the sufficient assumptions guaranteeing the existence and the other results stated on the unique equilibrium measure. As a consequence, we obtain a sufficient condition for the billiard flow to admit a unique measure of maximal entropy, and give examples of billiard tables that satisfy this condition. Finally, we prove that this measure is Bernoulli, flow-adapted and has full support.



**Keywords:** Dynamical Systems, dispersive billiard, thermodynamic formalism, anisotropic Banach space, transfer operator, spectral theory, Ruelle resonance.

# Contents

# Chapter 1

## Introduction

The field of Dynamical Systems is a very broad branch of mathematics focused on the long term behaviour caused by some evolution law. In this chapter, we first motivate the statistical approach in the study of a transformation or flow. We then insist on the fact that not all invariant measures give the same amount of information, and we present some of the most important ones. Next, we focus on the particular case of hyperbolic dynamics, more precisely Anosov maps and flows, and we describe the properties of the above mentioned invariant measures. We also present various ways these measures can be constructed, in particular through a functional approach. This last method can also give asymptotic expansions from which we can deduce the rate of mixing. Finally, we present the contributions of this thesis.

## 1.1 Motivations

The idea that a dynamical system derived from classical mechanics is not subject to statistical properties goes back to Laplace and is based on the fact that the motion of such a system is uniquely determined once initial conditions are given. However, in practical terms, the initial conditions are never known with perfect accuracy, and it is therefore the motion of a neighbourhood of the initial condition, a *cell*, in the phase space that must be studied, where each point of the cell moves accordingly to given differential equations (of motion). More generally, one could be tempted to consider other flows than Hamiltonian ones, or even to consider discrete time dynamics through iteration of a map. This is the settings we will consider. We say that the trajectory of a point $x$ is stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all large enough time $t$, the image of the $\delta$-neighbourhood of $x$ by the time $t$ of the motion is contained in the $\varepsilon$-neighbourhood of the point $x_t$, image of $x$ after a time $t$. Clearly, the motion of a cell containing a point whose trajectory is stable, is well described by the motion of this point. Now, for some transformations – even for some conservative ones, that are very easy to describe, see Example 1.1.1 – cells having initially a regular form become distorted, take intricate form, and distribute themselves into complicated shapes in the phase space.

**Example 1.1.1.** *One of the most famous, and simplest, example of chaotic map is the so called* Arnold's cat map. *It is obtained by letting the matrix* $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ *act on the two-torus* $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ *As shown in Figure 1.1, the image after a single iteration of the*

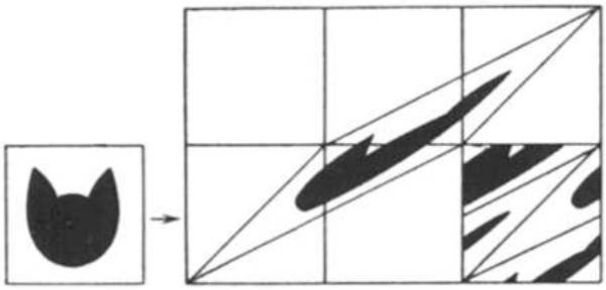

**Figure 1.1 –** The picture on the left represents a cell with the shape of a cat inside a fundamental domain of the torus. The picture on the right represents the action of $A$ on the fundamental domain in the plane, as well as the image of the cell inside a single fundamental domain (the image is taken from Arnold's book).

*initially cat-shaped cell no longer looks like a cat at all. One can easily imagine that the situation can only get worse with more iterations.*

Clearly, the spreading of those cells comes from instabilities, that is, arbitrarily close points eventually diverge and seem to move independently. We arrive at the idea that, for unstable motions, trajectories should present statistical properties, although they are deterministic.

### 1.1.1 Statistical description of orbits

By the statistical description of an orbit, we mean its asymptotic distribution. More precisely, given a continuous self map $T : X \to X$ of a metrizable space $X$, and a subset $U \subset X$, we are interested in the number of visits to the set $U$ under the first $n$ iterates of a point $x \in X$, that is, in the sequence

$$F_U(T,x)_n = \frac{\#\{i \in [0, n-1] \mid T^i(x) \in U\}}{n} = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_U \circ T^i(x)\,.$$

Instead of considering discontinuous observables, such as $\mathbb{1}_U$, it is preferable to consider continuous ones. Indeed, from the Riesz representation theorem, the set of Borel measures over $X$ can be identified to the topological dual of the continuous functions $(C^0(X))^*$. Furthermore, given $x \in X$, the map associating to each $\varphi \in C^0(X)$ its Birkhoff average $I_x(\varphi, n) = \frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ T^i(x)$ is linear. Thus, if for all $\varphi$, $I_x(\varphi, n)$ converges, then there exists a measure $\mu_x$ such that

$$\lim_{n\to\infty} I_x(\varphi, n) = \int \varphi \, \mathrm{d}\mu_x$$

and since $I_x(1, n) = 1$ for all $n$, we get that $\mu_x(X) = 1$, that is $\mu_x$ is actually a probability measure. Notice that since $n(I_x(\varphi \circ T, n) - I_x(\varphi, n))$ is bounded, the limits (when they exist) associated to $\varphi \circ T$ and $\varphi$ coincide, thus for any Borel set $A \subset X$, $T_*\mu_x(A) := \mu_x(T^{-1}A) = \mu_x(A)$, that is $\mu_x$ is a $T$-invariant measure.

Two natural questions appear:

i) Does such a point $x$ exist?

ii) If $\mu$ is a $T$-invariant measure, is there a point $x$ such that $\mu_x = \mu$?

To give a positive answer, we proceed as follows: from the Krylov–Bogolubov theorem, there exists a $T$-invariant measure $\mu$. Using the Birkhoff ergodic theorem, for all $\varphi \in \mathrm{L}^1(X, \mu)$, $I_x(\varphi, n)$ admits a limit for $\mu$-almost every $x$. Since $X$ is compact, there exist a sequence $(\varphi_i)_{i\in\mathbb{N}}$ of continuous functions that is dense in $C^0(X)$. Let $x$ be such that $I_x(\varphi_i, n)$ converges for all $i$. Then, for any $\varphi \in C^0(X)$, $I_x(\varphi, n)$ is a Cauchy sequence, and hence converges. Now for the second question, we repeat the same construction starting from an ergodic measure $\mu$, where a measure is said to be ergodic if it is irreducible in the sense that for all $T$-invariant Borel set $A$, $\mu(A)$ is either 0 or 1. Ergodic measures are extremal points of the set of invariant measures, and thus always exist. In this case, the limit of $I_x(\varphi, n)$ is equal to $\int \varphi \, \mathrm{d}\mu$ for $\mu$-almost every $x$.

The fact that both i) and ii) have positive answers justifies the statistical approach. If $\mu$ is a $T$-invariant measure, the above discussion on the chaotic evolution of a cell can be quantified: $\mu$ is said to be mixing if for all Borel sets $A$ and $B$, then $\mu(A \cap T^{-n}B)$ converges to $\mu(A)\mu(B)$ when $n$ goes to infinity. In other words the cell $B$ spreads evenly in the phase space, according to the measure $\mu$.

## 1.2 Specific invariant measures and Entropy

Since a $T$-invariant measure is a fixed point of $T_*$, these measures are also important in the study of $T_*$. We are thus trading a nonlinear problem in finite dimension, with a linear problem in infinite dimensions: the action of the transfer operator $T_*$ on the Banach space of measures $(C^0(X))^*$. Since $T_*$ preserves the subset of probability measures $\mathcal{M}(X)$, we will restrict our attention to this subset. Furthermore, according to the previous section, the relevant probability measures in the description of the behaviour of $T$ are those that are $T$-invariant.

From now on, by measure we mean probability measure. If $\mu$ is a $T$-invariant measure we define

$$\mathcal{A}(\mu) := \{\nu \in \mathcal{M}(X) \mid \nu \ll \mu\}.$$

The statistical behaviour of $\mu$ is related to the behaviour of $T_*$ on $\mathcal{A}(\mu)$ as follows. The measure $\mu$ is ergodic if and only if $\mu$ is the only fixed point of $T_*|_{\mathcal{A}(\mu)}$, while $\mu$ is mixing if and only if $\mu$ is an attractive fixed point of $T_*|_{\mathcal{A}(\mu)}$.

Now, not all invariant measures are relevant in the study of $T$. For example, if $T$ admits a periodic point $x$ of period $n$, then the measure $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i(x)}$ is $T$-invariant (and in fact ergodic) and describes perfectly the behaviour of the point $x$, but it gives very little information on the rest of the phase space (except if $x$ is an attractive fixed point).

### 1.2.1 Smooth invariant measures and physical measures

In classical mechanics, the differential equations of motion can be integrated into a flow which preserves a volume measure. Many other natural flows or transformations are in this situation, and this smooth measure is often the most studied one. Notice that the ergodic theorem then holds Lebesgue-almost everywhere. If such a measure is ergodic, it is then the unique invariant measure equivalent to Lebesgue.

Nonetheless, some systems are sometimes deprived of smooth measures. It is then tempting to find whether there exist measures with similar properties one expects from a smooth measure. Physical measures are those for which the ergodic theorem holds on a set of positive Lebesgue measure.

### 1.2.2 Entropy and Measures of maximal entropy

In 1958, taking inspiration from Shannon information theory, Kolmogorov [Kol58] introduced a quantity associated to each invariant measure: the Kolmogorov–Sinai entropy $h_\mu(T)$. This quantity is particularly important for many reasons, one is that the entropy is a conjugacy invariant. We recall briefly the definition of $h_\mu(T)$. Given a finite partition $\xi = \{A_1, \ldots, A_n\}$ of $X$ into measurable sets, define its static entropy by

$$H_\mu(\xi) = -\sum_{A\in\xi} \mu(A) \log \mu(A).$$

If $\xi_1$ and $\xi_2$ are two finite partitions, define the join partition $\xi_1 \vee \xi_2$ to be the partition of $X$ into sets of the form $A \cap B$, where $A \in \xi_1$ and $B \in \xi_2$. Since $T^{-1}\xi$ is a finite partition whenever $\xi$ is a finite partition, define $\xi_n = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-n+1}\xi$. Then the sequence $\log H_\mu(\xi_n)$ is subadditive, and therefore $\frac{1}{n} \log H_\mu(\xi_n)$ converges to a limit called $h_\mu(T, \xi)$. Finally, define the entropy of $\mu$ to be

$$h_\mu(T) := \sup\{h_\mu(T, \xi) \mid \xi \text{ is a finite partition into measurable sets}\}.$$

Morally, this quantity describes the complexity of $T$ perceived by $\mu$. In this sense, it is therefore natural to investigate the measures with maximal entropy, that is measures $\mu_{\text{MME}}$ such that $h_{\mu_{\text{MME}}}(T) = \sup\{h_\mu(T) \mid \mu \in \mathcal{M}(X),\, T_*\mu = \mu\}$.

When $T$ is a continuous map and $X$ is compact, then the following equality holds

$$h_{\text{top}}(T) = \sup\{h_\mu(T) \mid \mu \in \mathcal{M}(X),\, T_*\mu = \mu\}$$

and is called variational principle. Here $h_{\text{top}}(T)$ is the topological entropy of $T$, and is equal to the pressure (see the next subsection) of the zero potential.

Unfortunately, existence of such measures $\mu_{\text{MME}}$ is far from being automatic. Indeed, although $\mathcal{M}(X)$ is a compact set, the map $\mu \mapsto h_\mu(T)$ is usually not continuous. Nonetheless, in 1972, Bowen proved that if $T$ is expansive, that is, if

$$\exists \varepsilon > 0\, \forall x,\, y \in X \Big[ d(T^i(x), T^i(y)) < \varepsilon\,, \forall i \in \mathbb{Z} \Rightarrow x = y \Big]\,,$$

then $\mu \mapsto h_\mu(T)$ is upper-semicontinuous [Bow72a]. This regularity is sufficient to ensure the existence of measures of maximal entropy. Still, ergodicity does not insure the uniqueness as in the case of smooth measures (in fact, when $T$ is continuous and the set of measures of maximal entropy is not empty, at least one of those measures must be ergodic).

In 1974, Bowen introduced the specification property [Bow75] and proved that, in addition with expansiveness, it ensures the uniqueness of the measure of maximal entropy.

We now give some explicit examples of dynamics, either maps of flows, for which the measure of maximal entropy exists, is unique and is clearly identified.

**Example 1.2.1.** *Let $m \geqslant 2$ be an integer. Define $E_m$ on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ by*

$$E_m(x) = mx \bmod 1.$$

*The topological entropy of $E_m$ is equal to $\log m$. Furthermore, the Lebesgue measure is $E_m$-invariant, and its entropy is $\log m$. This is the only measure of maximal entropy of $E_m$.*

**Example 1.2.2.** *Let $\mathcal{A}$ be a finite set. We call $\mathcal{A}$ an alphabet. Define the set of bi-infinite words $\Omega = \mathcal{A}^{\mathbb{Z}}$ and the shift $\sigma : \Omega \to \Omega$ by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. Then for any stochastic matrix $M$ of size $\#\mathcal{A}$ and any vector $\nu$ with positive coefficients $\nu_i$ such that $\nu^T M = \nu^T$ and $\sum_i \nu_i = 1$, one can construct a $\sigma$-invariant measure $\mu_{P,\nu}$, called a Markov measure. By a simple computation, one get $h_{\mu_{P,\nu}}(\sigma) = -\sum_{i,j} \nu_i M_{i,j} \log M_{i,j}$. This quantity is maximized in the special case $\nu_i = M_{i,j} = 1/\#\mathcal{A}$, for all $i, j \in \mathcal{A}$. In this case $\mu_{P,\nu} = \nu^{\otimes\mathbb{Z}}$ – where $\nu$ is seen as a measure on $\mathcal{A}$ – and its entropy is equal to $\log \#\mathcal{A}$, which coincides with the topological entropy of $\sigma$. One can prove that this measure is the unique measure of maximal entropy.*

**Example 1.2.3.** *Given a matrix $A$ of size $n \times n$, with $n = \#\mathcal{A}$, whose coefficients $A_{ij}$ are in $\{0, 1\}$, define the subshift of finite type to be $\sigma$ restricted to the invariant subset*

$$\Omega_A = \{(x_i)_{i\in\mathbb{Z}} \mid \forall i \in \mathbb{Z},\, A_{x_i x_{i+1}} = 1\}.$$

*Denote the restriction of $\sigma$ to $\Omega_A$ by $\sigma_A$. If there exists $N$ such that every coefficient of $A^N$ is positive, then the topological entropy of $\sigma_A$ is equal to $\log \rho(A)$, where $\rho(A)$ is the spectral radius of $A$. Furthermore, there exists a unique measure $\mu_A$, called the Parry measure, with $h_{\mu_A}(\sigma_A) = \log \rho(A)$. This measure can be explicitly constructed from left and right eigenvectors of $A$ associated to the eigenvalue $\rho(A)$.*

**Example 1.2.4.** *As in Example 1.1.1, one can construct hyperbolic automorphisms of $\mathbb{T}^2$ from any matrix $A \in M_n(\mathbb{Z})$ with determinant $\pm 1$ and trace strictly larger than 2 (in absolute value). In this case, such $A$ has two distinct real eigenvalues $\lambda > 1$ and $\lambda^{-1}$. One can compute the topological entropy of the map induced by $A$ to be equal to $\log \lambda$. One can also prove that the Lebesgue measure is invariant and has entropy also equal to $\log \lambda$. It is the only measure of maximal entropy.*

**Example 1.2.5.** *In the case of the geodesic flow on a compact surface of constant negative curvature, the volume measure coincides with the measure of maximal entropy.*

### 1.2.3 Pressure and Equilibrium measures

In the case of symbolic dynamics, and later for continuous transformations, Ruelle introduced in 1972 [Rue73] a quantity generalizing the notion of topological entropy: the (topological) pressure. This quantity has then been studied in the general case by Walters [Wal75]. We recall briefly the definition of the pressure $P_*(T, g)$ associated to a potential $g : X \to \mathbb{R}$.

First, define the Bowen dynamical distance $d_n$ to be such that for all $x$ and $y \in X$,

$$d_n(x, y) = \max_{0 \leqslant i \leqslant n} d(T^i(x), T^i(y)) \,.$$

Given some $\varepsilon > 0$, we say that a set $E \subset X$ is $(n, \varepsilon)$ separated if for all distinct points $x$ and $y \in E$, $d_n(x, y) > \varepsilon$. Define the Birkhoff sum of $g$ to be $S_n g = \sum_{i=0}^{n-1} g \circ T^i$, and

$$P_*(T, g, \varepsilon, n) := \sup\{\sum_{x \in E} e^{S_n g(x)} \mid E \text{ is } (n, \varepsilon) \text{ separated}\},$$
$$P_*(T, g, \varepsilon) := \limsup_{n \to \infty} \frac{1}{n} \log P_*(T, g, \varepsilon, n),$$
$$P_*(T, g) := \lim_{\varepsilon \to 0} P(T, g, \varepsilon),$$

where the last limit exists because $\varepsilon \mapsto P_*(T, g, \varepsilon)$ is nonincreasing (the limit could be $\infty$). Define the topological pressure of $T$ under the potential $g$ to be $P_*(T, g)$. This quantity satisfies two remarkable results.

**Theorem 1.2.6.** *[Wal82, Theorems 9.10, 9.11] Assume that $T : X \to X$ is a continuous map on a compact metrizable set $X$. Then*
*i) $P_*(T, \cdot)$ determines the set of invariant measures $\mathcal{M}(X, T)$: if $\mu$ is a finite, signed measure, then $\mu \in \mathcal{M}(X, T)$ iff for all $g \in C^0(X)$, $\int_X g \, \mathrm{d}\mu \leqslant P_*(T, g)$;*
*ii) for all continuous $g$, $P_*(T, g) = P(T, g) := \sup\{h_\mu(T) + \int g \, \mathrm{d}\mu \mid \mu \in \mathcal{M}(X, T)\}$.*

In analogy with the case $g = 0$, ii) is also called variational principle, and the measures (if they exist) achieving the sup are called equilibrium measures (or equilibrium states). Here again, the existence of such measures is not always guaranteed. However, using again Bowen's results, if $T$ is expansive, then $\mu \mapsto h_\mu(T) + \int g \, \mathrm{d}\mu$ is upper-semicontinuous (since the first term is, and the second term is continuous by definition of the weak-$^*$ topology), thus, there exist equilibrium states.

In the next section, we will see that, for some transformations $T$, all the above mentioned invariant measures are equilibrium states.

## 1.3 The case of the Hyperbolic Dynamic

Particularly important and extensively studied dynamical systems are the family of hyperbolic dynamical systems. The interest in these dynamics goes back at least to the work of Hadamard [Had98] on the geodesic flow on negatively curved surfaces. A crucial point in the history of their study is the axiomatic definition given by Anosov of the flows and diffeomorphisms that now bear his name. The introduction of this definition was motivated by the study of the dynamical properties of the geodesic flow on the unit cotangent bundle of a Riemannian manifold of negative (a priori non-constant) sectional curvature. Research in this area has subsequently been very active and, although there are still unanswered questions, the understanding of hyperbolic dynamics has greatly improved since Anosov's early work, in particular through the development of many tools. Among these, we can mention Markov partitions, coupling arguments, Young towers, etc. One approach that has been particularly developed in recent decades is the one using functional analysis. This approach is particularly suitable for generalizations for dynamics whose hyperbolicity is weaker than the one defined by Anosov. It is from this approach that the results presented in this thesis are derived.

We start by recalling the definition of an Anosov diffeomorphism

**Definition 1.3.1.** *Let $M$ be a compact manifold and $T : M \to M$ be a $C^1$ diffeomorphism. We say that $T$ is an Anosov diffeomorphism if, for every $x \in M$ there is a splitting of the tangent space of $M$ at $x$*

$$T_xM = E^u_x \oplus E^s_x\,,$$

*and there are constants $C > 0$, $\Lambda > 1$ and a smooth Riemannian metric on $M$ such that*

*(i) for every $x \in M$, and $* \in \{s, u\}$, we have $D_xT(E^*_x) = E^*_{T(x)}$;*

*(ii) for every $x \in M$, $v \in E^u_x$ and $n \in \mathbb{N}$, we have $|D_xT^{-n}(v)| \leqslant C\Lambda^{-n}|v|$;*

*(iii) for every $x \in M$, $v \in E^s_x$ and $n \in \mathbb{N}$, we have $|D_xT^{n}(v)| \leqslant C\Lambda^{-n}|v|$.*

An example of such diffeomorphism is given in Example 1.1.1. Actually, any matrix $A \in SL(2, \mathbb{Z})$ with no eigenvalue of modulus 1 induces an Anosov diffeomorphism on the torus $\mathbb{R}^2/\mathbb{Z}^2$, as in Example 1.2.4.

The definition of Anosov flows is obtained by modifying the above definition as follows: a flow $(\phi_t)_{t\in\mathbb{R}}$, generated by a zero-free vector field $X$, is said to be Anosov if for each $x \in X$ there is a splitting $T_xM = E^0_x \oplus E^u_x \oplus E^s_x$, such that $E^0_x$ is the span of $X(x)$, and $E^u_x$, $E^s_x$ respectively satisfy (ii) and (iii) with $T$ replaced by $\phi_1$, and $n$ by $t \geqslant 0$.

### 1.3.1 The Ruelle–Perron–Frobenius transfer operator

There are many different approaches to the construction of the measures discussed in Subsection 1.2. In the case of smooth invariant measures in the setting of Anosov maps, the first construction was performed by Sinai, Bowen and Ruelle [Bow08]. For this reason, these measures are called SRB measures. Their construction starts by proving the existence of a finite Markov partition, and of a (Lipschitz) semiconjugacy map between the hyperbolic diffeomorphism and a subshift of finite type. The next step is to exploit the fact that the SRB measure is the equilibrium state associated to the potential $g = -\log DT|_{E^u}$. In the uniformly hyperbolic case, $g$ is at least Hölder continuous by the theory of Hirsch–Pugh–Shub [HPS77]. Lifting this weight to the subshift of finite type produces a Hölder potential. The results on transfer operators developed in the case of symbolic dynamics yield an equilibrium state, which is exponentially mixing for Hölder observables if the subshift is topologically mixing. The drawback of this method is that a lot of information is lost while going to the symbolic setting (the maximal smoothness there is only Lipschitz).

Actually, one could avoid the coding step by considering directly the action of the transfer operator $T_*$. The construction of an SRB measure from this method arises from the following heuristics. Since for each $\mu = \rho\lambda \in \mathcal{A}(\lambda)$, we have $T_*(\rho\lambda) = \left(\frac{\rho}{JT} \circ T^{-1}\right)\lambda \in \mathcal{A}(\lambda)$, if $T$ admits an SRB measure which is equivalent to the Lebesgue measure $\lambda$, there is some $\rho$ such that $\frac{\rho}{JT} \circ T^{-1} = \rho$. It is then natural to restrict the action of $T_*$ to $\mathcal{A}(\lambda)$, considering instead the so called Ruelle–Perron–Frobenius operator $\mathcal{L} : L^1(M, \lambda) \to L^1(M, \lambda)$ defined by

$$\mathcal{L}(\rho) = \frac{\rho}{JT} \circ T^{-1}\,. \tag{1.3.1}$$

Notice that $||\mathcal{L}(\rho)||_{L^1(M,\lambda)} = ||\rho||_{L^1(M,\lambda)}$. Now, if there exists a nonnegative $\rho$ such that $\mathcal{L}(\rho) = \rho$, normalized so that $\int \rho \, \mathrm{d}\lambda = \lambda(\rho) = 1$, then the measure $\mu$ defined by

$\mu_{\mathrm{SRB}}(\varphi) = \frac{\lambda(\varphi\rho)}{\lambda(\rho)}$ is $T$-invariant since

$$\begin{aligned}\lambda(\rho)(T_*\mu)(\varphi) &= \int \varphi \circ T \rho \,\mathrm{d}\lambda = \int \varphi \circ T \rho \,\mathrm{d}(\mathcal{L}^*\lambda) = \int \mathcal{L}(\varphi \circ T\rho)\,\mathrm{d}\lambda \\ &= \int \varphi \mathcal{L}(\rho)\,\mathrm{d}\lambda = \int \varphi\rho \,\mathrm{d}\lambda = \lambda(\rho)\mu(\varphi)\end{aligned}$$

where in the second equality we used that $\lambda$ is a left eigenvector of $\mathcal{L}$ associated to the eigenvalue 1 (which is a consequence of the change of variable formula). In this sense, we have paired left and right eigenvectors associated to the maximal eigenvalue of $\mathcal{L}$ in order to construct the invariant measure $\mu_{\mathrm{SRB}}$. This method of constructing invariant measure by pairing eigenvectors will be used in the next subsection where the operator $\mathcal{L}$ will be equipped with a different weight.

Furthermore, since

$$\int \varphi \circ T^n \psi \,\mathrm{d}\mu - \int \varphi \,\mathrm{d}\mu \int \psi \,\mathrm{d}\mu = \int \varphi\Big(\mathcal{L}^n(\psi\rho) - \rho \int \psi\rho \,\mathrm{d}\lambda\Big)\,\mathrm{d}\lambda,$$

the rate of mixing is governed by the decay to zero of $\mathcal{L}^n(\phi) - \rho \int \phi \,\mathrm{d}\lambda$. When 1 is a simple eigenvalue of $\mathcal{L}$, then $\phi \mapsto \rho \int \phi \,\mathrm{d}\lambda$ is the spectral projection to the eigenspace spanned by $\rho$. Therefore, the spectral theory of $\mathcal{L}$ also gives information on the rate of mixing of $\mu_{\mathrm{SRB}}$.

It turns out that finding such an eigenvector $\rho$ is usually not that easy and some more involved work has to be done. This issue have been much studied in the last decades and the solution essentially consists in introducing well chosen Banach spaces of distributions on which $\mathcal{L}$ acts (after being extended). Actually, there are many different constructions for those Banach spaces [BKL02, GL06, BCFT18, Bal18]. These constructions (almost all) rely on finding two *anisotropic* norms, a strong one $||\cdot||$ and a weak one $||\cdot||_w$, on $C^r(M,\mathbb{R})$ for some $r \in [1,+\infty]$. These norms are distributional norms that satisfy $||\cdot||_w \leqslant ||\cdot|| \leqslant ||\cdot||_{C^r}$. The strong norm is anisotropic, in the sense that, for $\varphi \in C^r(M,\mathbb{R})$, $||\varphi||$ measures the regularity (in a classical sense) of $\varphi$ in the stable directions, while it measures the regularity, in a distributional sense, of $\varphi$ in the unstable directions. In view of using functional analysis techniques, it is more convenient to work with Banach spaces. Therefore, let $\mathcal{B}$ and $\mathcal{B}_w$ be the completions of $C^r(M,\mathbb{R})$ with respect to the norms $||\cdot||$ and $||\cdot||_w$. These spaces are the ones on which we want to study the action of $\mathcal{L}$. To do so, we first need to extend the transfer operator onto these Banach spaces. A convenient way to do so is to find $||\cdot||$ and $||\cdot||_w$ so that

$$C^r(M,\mathbb{R}) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (C^r(M,\mathbb{R}))^*\,, \tag{1.3.2}$$

where the first two injections are the canonical maps, the second map is compact, and the third embedding is obtained by extending $\varphi \mapsto \varphi\lambda$. In this case, we can see the elements of $\mathcal{B}$ and $\mathcal{B}_w$ as distributions and extend $\mathcal{L}$ on $(C^r(M,\mathbb{R}))^*$ by setting

$$\mathcal{L}(f)(\psi) = \langle \psi \circ T, f\rangle\,, \quad \psi \in C^r(M,\mathbb{R}).$$

Notice that it is indeed an extension, since for $f \in C^r(M,\mathbb{R})$, we get

$$\mathcal{L}(f\lambda)(\psi) = \int f\psi \circ T \,\mathrm{d}\lambda = \int \frac{f}{JT} \circ T^{-1}\psi \,\mathrm{d}\lambda = \Big(\frac{f}{JT} \circ T^{-1}\lambda\Big)(\psi),$$

and by the identification $f \mapsto f\lambda$, $\mathcal{L}$ takes the form (1.3.1) on smooth function.

Every construction of anisotropic Banach spaces, in this context, has in mind that $\mathcal{L}$ should be quasi-compact, in the sense that,

**Definition 1.3.2.** *For a given bounded linear operator $\mathcal{L}$ from a Banach space $\mathcal{B}$ to itself, the essential spectral radius $r_{\text{ess}}(\mathcal{L})$ is the infimum of the $r > 0$ such that the intersection of the spectrum $\sigma(\mathcal{L})$ with the disc $\{z \in \mathbb{C} \mid |z| > r\}$ is comprised of finitely many eigenvalues with finite algebraic multiplicities. We say that $\mathcal{L}$ is quasi-compact if the essential spectral radius of $\mathcal{L}$ is strictly smaller than its spectral radius $r(\mathcal{L})$.*

A way to prove that $\mathcal{L}$ is quasi-compact is to exploit the weak space, and to show that $\mathcal{L}$ satisfies a *Lasota–Yorke type inequality* (also called *Doeblin–Fortet*, that is

**Definition 1.3.3.** *We say that the operator $\mathcal{L}$ satisfies the Lasota–Yorke inequality if there exist $0 < \theta < r(\mathcal{L})$ and constants $A$ and $B$ such that for all $n \geqslant 0$ and all $f \in \mathcal{B}$,*

$$||\mathcal{L}^n(f)|| \leqslant A\theta^n ||f|| + B||f||_w \, .$$

*The case where $\theta = r(\mathcal{L})$ and $B = 0$ should be thought as a degenerate case.*

According to the work of Hennion [Hen93], after a spectral formula due to Nussbaum [Nus70], if $\mathcal{L}$ satisfies such inequality for some $\theta < r(\mathcal{L})$, then $r_{\text{ess}}(\mathcal{L}) \leqslant \theta$ and thus $\mathcal{L}$ is quasi-compact.

First, the peripheral spectrum of $\mathcal{L}$ has to be investigated. It is made of finitely many eigenvalues $\lambda_1, \ldots, \lambda_K$ with modulus equal to the spectral radius of $\mathcal{L}$, with $\lambda_1 = 1$. Let $\Pi_i$ be the spectral projection onto the eigenspace associated to $\lambda_i$. These projectors are well defined operators from $\mathcal{B}$ to itself with finite dimensional ranges, and for all $f \in \mathcal{B}$, $\Pi_i(f)$ can be extended into a signed measure on $M$. In fact, the projector can be explicitly written as the limit of the averaged action of $\lambda_i^{-n}\mathcal{L}^n$. Letting $\mu = \Pi_1(1)$, one can show that all measures in the range of some $\Pi_i$ are absolutely continuous with respect to $\mu$. In fact, from the characterisation of $\Pi_1$, $\mu$ is the limit of $\frac{1}{n}\sum_{k=0}^{n-1}(T^k)_*\lambda$, which is another (equivalent) definition of the SRB measure, so that $\mu_{\text{SRB}} = \mu$. One can construct finitely many ergodic measures from a basis of the range of $\Pi_1$ such that they are the ones appearing in the ergodic decomposition of $\mu$. In particular, we get that $\mu$ is ergodic if and only if the range of $\Pi_1$ is one-dimensional. In the case of Anosov diffeomorphism, a sufficient condition for 1 to be a simple eigenvalue of $\mathcal{L}$ is that $T$ is topologically transitive. Moreover, $\mu$ is mixing if and only if the peripheral spectrum of $\mathcal{L}$ is reduced to the simple eigenvalue 1. In the case of an Anosov diffeomorphism, a sufficient condition for that is the topological mixing property of $T$.

As above, we can write $\mu$ as a pairing of left and right eigenvectors of $\mathcal{L}$. Indeed, let $e_1$ be the element of $(C^r(M, \mathbb{R}))^{**}$ defined by $e_1(f) = \langle 1, f\rangle$ for $f \in (C^r(M, \mathbb{R}))^*$. We get that

$$\mathcal{L}^*(e_1)(f) = \langle f, \mathcal{L}^* e_1\rangle = \langle \mathcal{L}f, e_1\rangle = \langle 1, \mathcal{L}f\rangle = \langle 1, f\rangle = e_1(f).$$

Hence $e_1$ is a left eigenvector of $\mathcal{L}$ associated to the eigenvalue 1. Pairing the left and right eigenvectors of $\mathcal{L}$ associated to 1, we get

$$\frac{e_1(\varphi\mu)}{e_1(\mu)} = \frac{\mu(\varphi)}{\mu(1)} = \mu(\varphi)\,, \quad \varphi \in C^0(M, \mathbb{R}).$$

In the case where $\mu$ is mixing, the rest of the spectrum of $\mathcal{L}$ gives rise to an asymptotic expansion of the correlation functions. Actually, in [BKL02, GL06, BCFT18, Bal18] it is not only two Banach spaces that are constructed but an infinite family (ordered by smoothness), giving better estimates on the essential spectral radius of $\mathcal{L}$ the smoother $T$ is. More precisely, if $T$ is chosen to be $C^\infty$, we can find Banach spaces so that the constant $\theta$ from the Lasota–Yorke inequality is arbitrarily small. In other words, given any $\varepsilon > 0$, there is a Banach space $\mathcal{B}$ such that $r_{\mathrm{ess}}(\mathcal{L}) < \varepsilon$, and thus, letting $(\gamma_j)_{1\leqslant j\leqslant D}$ be the distinct eigenvalues of $\mathcal{L}$ of modulus larger than $\varepsilon$, with $\gamma_1 = 1$, there exists $\kappa \geqslant 1$ such that we can write

$$\mathcal{L}(\varphi) = \sum_{j=1}^{D} (\gamma_j \mathrm{Id} + \mathcal{N}_j)\Pi_j(\varphi) + \mathcal{R}(\varphi)\,, \quad \varphi \in \mathcal{B}\,, \tag{1.3.3}$$

where $\mathcal{R}$ has a spectral radius smaller than $\varepsilon$, the $\Pi_j$ are finite rank projections ($\Pi_j\Pi_k = \delta_{jk}\Pi_j$), and the $\mathcal{N}_j$ are finite rank operators such that $\Pi_j\mathcal{N}_k = \mathcal{N}_k\Pi_j = \delta_{jk}\mathcal{N}_k$ and $(\mathcal{N}_j)^\kappa = 0$ (nilpotence). In addition,

$$\Pi_j\mathcal{R} = \mathcal{R}\Pi_j = \mathcal{N}_j\mathcal{R} = \mathcal{R}\mathcal{N}_j = 0\,, \quad \mathcal{N}_j\mathcal{N}_k = \delta_{jk}(\mathcal{N}_j)^2.$$

Thus, we get that

$$\mathcal{L}^n(\varphi) = \sum_{j=1}^{D} (\gamma_j \mathrm{Id} + \mathcal{N}_j)^n\Pi_j(\varphi) + \mathcal{R}^n(\varphi) = \sum_{j=1}^{D} \gamma_j^n \left(\sum_{l=0}^{\kappa} \binom{n}{l} \gamma_j^{-l}\mathcal{N}_j^l\right)\Pi_j(\varphi) + \mathcal{R}^n(\varphi).$$

Since we assumed that $\gamma_1 = 1$ is simple, $\mathcal{N}_1 = 0$ and $\Pi_1(\varphi) = e_1(\varphi)\mu$. Thus, for any $\varphi$ and $\psi \in C^r(M, \mathbb{R})$, with $r$ large enough, we get

$$\begin{aligned}
&\Big|\int \varphi\circ T^n \psi \,\mathrm{d}\mu - \int \varphi \,\mathrm{d}\mu \int \psi \,\mathrm{d}\mu - \sum_{j=2}^{D} \gamma_j^n \left(\sum_{l=0}^{\kappa} \binom{n}{l} \gamma_j^{-l}\mathcal{N}_j^l\right)\Pi_j(\psi\mu)(\varphi)\Big| \\
&\qquad = \Big|\langle \varphi\circ T^n, \psi\mu\rangle - e_1(\psi\mu)\mu(\varphi) - \sum_{j=2}^{D} \gamma_j^n \left(\sum_{l=0}^{\kappa} \binom{n}{l} \gamma_j^{-l}\mathcal{N}_j^l\right)\Pi_j(\psi\mu)(\varphi)\Big| \\
&\qquad = \Big|\langle \varphi, \mathcal{L}^n(\psi\mu)\rangle - \Pi_1(\psi\mu)(\varphi) - \sum_{j=2}^{D} \gamma_j^n \left(\sum_{l=0}^{\kappa} \binom{n}{l} \gamma_j^{-l}\mathcal{N}_j^l\right)\Pi_j(\psi\mu)(\varphi)\Big| \\
&\qquad = \Big|\mathcal{L}^n(\psi\mu)(\varphi) - \sum_{j=1}^{D} \gamma_j^n \left(\sum_{l=0}^{\kappa} \binom{n}{l} \gamma_j^{-l}\mathcal{N}_j^l\right)\Pi_j(\psi\mu)(\varphi)\Big| \\
&\qquad = \Big|\mathcal{R}^n(\psi\mu)(\varphi)\Big| \leqslant C|\varphi|_{C^r}|\psi|_{C^r}||\mu||\,\varepsilon^n.
\end{aligned} \tag{1.3.4}$$

In other words, the spectral theory of $\mathcal{L}$ gives an asymptotic expansion of the correlation between $\varphi$ and $\psi$.

Finally, one can prove that $\mu$ is the equilibrium state associated to the potential $-\log J^uT$, by using the operator $\mathcal{L}$ in order to get a sharp upper bound on the volume of Bowen balls of small radius, involving the Birkhoff sum of the potential as well as its pressure. Using Brin–Katok's theorem, we relate these volume estimates to the entropy of $\mu$, proving that $\mu$ is such that $P(T, -\log J^uT) = h_\mu(T) - \int \log J^uT \,\mathrm{d}\mu$.

### 1.3.2 Weighted transfer operators

By analogy with the case of symbolic dynamic (see e.g. [Bow08]) where the equilibrium state of a potential $\phi$ is constructed from the pairing of left and right eigenvectors of the weighted transfer operator

$$(\underline{\mathcal{L}}_\phi f)(\underline{x}) = \sum_{\underline{y}\in\sigma^{-1}\underline{x}} e^{\phi(\underline{y})} f(\underline{y})$$

we wish to do the same directly for $T$. We then define the weighted transfer operator, with weight $g$,

$$\tilde{\mathcal{L}}_g(f) := \left(e^g J^u T \frac{f}{JT}\right) \circ T^{-1}\,, \quad f \in C^r(M,\mathbb{R})\,.$$

The unstable Jacobian appears here so that for $g = -\log J^u T$, we recover the operator from the previous section. Yet, this operator can be slightly simplified since $JT(x) = J^s T(x) J^u T(x) \frac{E\circ T(x)}{E(x)}$, where $E(x)$ is the sin of the angle between the stable and the unstable bundles $E^s$ and $E^u$ at $x$. Then, replacing $g$ by $g - \log E \circ T + \log E$, which are cohomologous and should give rise to the same equilibrium states, we finally define

$$\mathcal{L}_g(f) := \left(e^g \frac{f}{J^s T}\right) \circ T^{-1}\,, \quad f \in C^r(M,\mathbb{R})\,. \tag{1.3.5}$$

The principal problem here is that, for smooth potential $g$, the function $1/J^s T$ is not smooth. The initial solution provided by Gouëzel and Liverani [GL08] was slightly different. They still consider a weighted transfer operator acting on an anisotropic space obtained as a completion, however, the space to be completed is radically different from a space of smooth functions. Indeed, they considered the space of $C^{r-1}$ sections of the line bundle over $\mathcal{G}$, where $\mathcal{G}$ is the Grassmannian of the oriented $d_s$-dimensional subspace of the tangent bundle $TM$, with $d_s$ the dimension of the stable bundle $E^s$. The transfer operator they used also has a weight, but there is no $J^s T$ in it. The rest of their analysis also consists in proving the Lasota-Yorke inequality, and then to study the peripheral spectrum. Pairing left and right eigenvectors associated to the eigenvalue equal to the spectral radius gives rise to an invariant measure. This measure is proved to be the expected equilibrium state by controlling the measure of Bowen balls.

In dimension two, another way to bypass this difficulty is provided by Demers [Dem21] and consists in making use of the SRB measure. For now, only the measure of maximal entropy, corresponding to $g = 0$, has been constructed, but it might be possible to adapt the construction to more general potential $g$ through heavier computations. The starting point is to replace the identification $f \mapsto f\lambda$ by $f \mapsto f\mu_{\mathrm{SRB}}$, so that the extension of $\mathcal{L}_0$ to the dual is formally

$$\mathcal{L}_0(f)(\psi) = \left\langle \frac{\psi\circ T}{J^s T}, f \right\rangle.$$

The spaces $\mathcal{B}$ and $\mathcal{B}_w$ are then obtained by completing $C^1(M,\mathbb{R})$ with respect to norms $||\cdot||$ and $||\cdot||_w$. The choice of these norms leads to the embedding (1.3.2) (where the second one is compact), except for the last one where the dual of $C^1(M,\mathbb{R})$ must be replaced by the dual of $C^\alpha(\mathcal{W}^s)$, the space of functions which are $\alpha$-Hölder along pieces of stable manifolds. In this setting, $\mathcal{L}_0$ can be extended to operators from $\mathcal{B}$ to itself, as well as from $\mathcal{B}_w$ to itself. Furthermore, $\mathcal{L}_0$ satisfies Lasota–Yorke type inequalities on both spaces (the one on $\mathcal{B}_w$ is of the degenerated type). As is now usual, an invariant measure $\mu_0$ is obtained by pairing left and right eigenvectors of $\mathcal{L}_0$. Thanks to this particular structure, the $\mu_0$-measure of Bowen balls is sharply controlled, which in particular implies that $\mu_0$ is a measure of maximal entropy. Furthermore, uniqueness of such maximal measure is proven, as well as a spectral gap for $\mathcal{L}_0$. Thanks to this gap, a similar expansion as in (1.3.4) gives exponential mixing for $C^1$ observables.

It has to be noted that the construction from [Dem21] was done in the more general context of piecewise hyperbolic maps (in dimension two) with bounded derivative.

## 1.4 Main results, in contexts

This thesis is essentially divided into two parts (of unequal length). The first one is devoted to an alternative proof of the absence of the deviations of the ergodic integrals of Giulietti–Liverani flows, while the second part is devoted to construct equilibrium states – and in particular the measure of maximal entropy – for dispersive billiard flows.

### 1.4.1 Absence of Deviations for parabolic flows

In their paper [GL19], Giulietti and Liverani introduced a flow $h^t$ obtained by integrating the one-dimensional stable foliation of an Anosov diffeomorphism $F$ of the two-dimensional torus $\mathbb{T}^2$. Similar flows have already been introduced in the past, and it is known since the work of Furstenberg that the classical horocycle flow (associated to the geodesic flow on a compact negatively curved surface) is uniquely ergodic. Using symbolic dynamics, Furstenberg results have been extended by Marcus [Mar75a, Mar75b] to flows generated by one-dimensional unstable foliation of an Anosov diffeomorphism or flow, and then with Bowen [BM77] to higher dimensional foliation.

Giulietti and Liverani give a new proof that $h^t$ is uniquely ergodic – we call the unique invariant measure $\mu^s$. Then, they also show that $h^t$ is minimal and admits a transversal curve such that the first return map has a rotation number of constant type. For a given $C^r$ Anosov diffeomorphism $F$, Giulietti and Liverani introduce a suitable Banach space $\tilde{\mathcal{B}}_{\mathrm{GL}}$, on which acts the transfer operator $\tilde{\mathcal{L}}$ associated to $F$. For large enough $r$, they provide an asymptotic expansion of

$$H_{x,T}(f) := \int_0^T f(h^t(x))\,\mathrm{d}t\,, \quad x \in \mathbb{T}^2,\, f \in C^r(\mathbb{T}^2, \mathbb{C}), \tag{1.4.1}$$

from eigenvectors of the dual operator $\tilde{\mathcal{L}}^*$, associated to eigenvalues $\{\tilde{\rho}_j\}_{j=0}^{N_{\mathrm{GL}}}$ of modulus strictly larger than the essential spectral radius $\tilde{\rho}_{\mathrm{GL}}$. The $\tilde{\rho}_j$'s are called *Ruelle resonances*, and those of modulus strictly larger than 1 are called deviation resonances. The dominant term of the expansion is given by $T\mu^s(f)$, corresponding to the trivial deviation resonance $\tilde{\rho}_0 = e^{h_{\mathrm{top}}}$, where $h_{\mathrm{top}}$ is the topological entropy of $F$. Furthermore, the error term of the expansion is a negative power law.

In order to fix the ideas, we state the expansion in the simpler case where there are no Jordan blocks (as in [Bal22, Eq. (1.2)]): For any $\delta > 0$ there are a constant $C$ and $\{C_j(x,T)\}_{j=1}^{N_{\mathrm{GL}}}$ with $\sup_{x,T,j} |C_j(x,T)| \leqslant C$, such that for all $f \in C^r(\mathbb{T}^2, \mathbb{C})$,

$$H_{x,T}(f) = T\mu^s(f) + \sum_{j=1}^{N_{\mathrm{GL}}} T^{\theta_j} C_j(x,T) \mathcal{O}_j(f \circ \pi) + \mathcal{R}_{x,T}(f)\,,$$

where $\theta_j = \frac{\log|\tilde{\rho}_j|}{h_{\mathrm{top}}} < 1$, $\mathcal{O}_j \in \tilde{\mathcal{B}}^*_{\mathrm{GL}}$ is an eigenvector of $\tilde{\mathcal{L}}^*$ associated to the eigenvalue $\tilde{\rho}_j$, and the rest satisfies

$$\sup_x |\mathcal{R}_{x,T}(f)| \leqslant C\Big(T^{\theta_{\min}}||f||_{C^r} + \sup|f|\Big), \quad \theta_{\min} = \frac{\log \tilde{\rho}_{\mathrm{GL}} + \delta}{h_{\mathrm{top}}} < 0,$$

and $\pi$ is the projection from the unit tangent bundle of $\mathbb{T}^2$ to $\mathbb{T}^2$.

Recently, Baladi [Bal22] and Forni [For22] provided independent proofs of the absence of deviation resonances in the general case (with possibly Jordan blocs). Their proofs are quite different: Baladi showed that $F$ does not have non trivial deviation resonance using methods derived from dynamical determinants, while Forni used the action of the (pseudo-) Anosov $F$ on the first cohomology and proved that deviation resonances do not exist on surfaces of genus one.

In Chapter 2, we give a short proof of a result implying the absence of deviation resonances for Giulietti–Liverani flows. Actually, this result gives a logarithmic bound on the growth of $H_{x,T}(f)$ for more general flows:

**Theorem 1.4.1.** *If $h_t$ is a $C^1$ flow on the torus $\mathbb{T}^2$ without critical points nor periodic orbits – in particular it admits a transversal curve $\gamma$ and is uniquely ergodic with invariant measure $\mu$ – and if the rotation number of the Poincaré first return map $R$ to $\gamma$ is of constant type, then there exist constants $K_1$ and $K_2$ such that for any $C^1$ observable $f$ with $\int f \,\mathrm{d}\mu = 0$, any $x$ and any $T > 0$,*

$$|H_{x,T}(f)| \leqslant K_1||f||_{C^1} \log(1 + T) + K_2||f||_{C^1}$$

Furthermore, in the second part of Chapter 2, we give an explicit construction of a $C^1$ flow $h_t$ on $\mathbb{T}^2$, renormalized by an Axiom A diffeomorphism $f_\beta$, satisfying the assumptions of Theorem 1.4.1. By a renormalization, we mean that $f_\beta \circ h_t = h_{\lambda^{-1}t} \circ f_\beta$, where here, $\lambda^{-1} < 1$ is the uniform contraction factor of $f_\beta$ associated to the stable foliation of its hyperbolic set. In particular, we are able to compute the rotation number of the first return map to a specific transversal section and we prove that it is a quadratic integer – and thus, of constant type.

### 1.4.2 Equilibrium states and Measure of maximal entropy for billiard flows

Chapters 3 and 4 are dedicated to the constructions of equilibrium states for the Sinai billiard flow, and more specifically the measure of maximal entropy.

Dispersing billiards, as introduced by Sinai [Sin70], form a class of hyperbolic dynamical systems with discontinuities and unbounded derivative at the singularities. It is then natural to try to adapt the methods used in the context of Anosov dynamics to those systems.

More precisely, a dispersing billiard – or (the quotient modulo $\mathbb{Z}^2$ of) two dimensional periodic Lorentz gaz – is a set $Q = \mathbb{T}^2 \setminus B$, where $B = \sqcup_{i=1}^D B_i$ for some integer $D$, and the $B_i$'s are disjoint closed domains, strictly convex with $C^3$ boundaries. The $B_i$'s are called scatterers. The billiard flow $\phi_t$ is the motion of a point particle travelling at unit speed on $Q$ and doing specular reflections off the boundary of the scatterers. By identifying the incoming collisions with the outgoing ones in $\Omega = Q \times \mathbb{S}^1$, $\phi_t$ is a continuous flow. Nonetheless, at grazing collisions – those tangential to a scatterer – the flow is not differentiable, its derivative is actually unbounded at those singularities.

Notice that the boundary of the scatterers, after identification, $M$ is a section for $\phi_t$, and when the first return time function $\tau$ is bounded $\phi_t$ is actually the suspension of the first return map $T$ to $M$ under the time $\tau$. The map $T$ is called the collision map, and is discontinuous at grazing collisions.

Since $\phi_t$ and $T$ are derived from models from classical mechanic, they both preserve some volume measures (which are SRB measures). It is by the mean of those measures that $\phi_t$ and $T$ have first been extensively studied. Those measures are ergodic, K-mixing [Sin70, BS73, SC87], and even Bernoulli [GO74, CH96]. They also have stronger statistical properties. Both are exponentially mixing [You98, DZ11, BDL18]. Chronologically, Young was the first to prove the exponential mixing for the SRB measure of $T$. It was through the development of a new technique: the Young towers. Only a year later, she introduced again a new technique, borrowed from the probability theory: coupling, and derived again the exponential mixing. Later on, Dolgopyat simplified this argument. Finally, Demers and Zhang [DZ11] contructed anisotropic Banach spaces on which the transfer operator associated to $T$ is quasi-compact and has a spectral gap. The exponential mixing of the SRB measure of the flow is a recent result [BDL18], which also relies on the construction of anisotropic Banach spaces of distributions.

Until very recently, only some perturbations of the SRB measure have been studied [CWZ17, DRBZ18], and not so much for other invariant measures.

Baladi and Demers [BD20] introduced Banach spaces such that the weighted transfer opertor $\mathcal{L}_0$ – weighted so that the measures of maximal entropy are expected to be obtained – satisfies (degenerate) Lasota–Yorke type inequalities. To do so, Baladi and Demers need two technical assumptions: the first one is that the billiard must have finite horizon in the sense that no orbit makes only grazing collisions – in particular, the return time $\tau$ is bounded. The second assumption quantifies the recurrence of the singular set: for $\varphi_0 \lesssim \pi/2$, we say that a collision is $\varphi_0$-grazing if the angle it makes with the normal to the scatterer is greater (in absolute value) than $\varphi_0$. For all $\varphi_0$ and $n_0 \geqslant 1$, define $s_0 = s_0(n_0, \varphi_0)$ to be the maximal frequency of $\varphi_0$-grazing collisions among $n_0$ consecutive collisions. The *sparse recurrence* assumption from [BD20, Eq. (1.5)] is then

$$\exists\, \varphi_0\,,\, n_0 \text{ such that } \quad h_* > s_0 \log 2,$$

where $h_*$ is a quantity which coincides with the topological entropy as defined by Bowen.

Although $\mathcal{L}_0$ is not a priori quasi-compact, Baladi and Demers managed to construct left and right eigenvectors of $\mathcal{L}_0$ associated to the eigenvalue $e^{h_*}$. They show that by pairing these vectors, one obtains a Radon measure $\mu_*$, that is K-mixing, Bernoulli, adapted and has maximal entropy. Finally, they prove that $\mu_*$ is the unique measure of maximal entropy of $T$.

The decay of correlation for $\mu_*$ have then been studied by Demers and Korepanov [DK22]. They prove that the mixing is, at least, polynomial for Hölder observables, as well as the Central Limit Theorem.

Using similar spaces as in [DZ11], Baladi and Demers [BD22] constructed equilibrium states $\mu_t$ for each potential $-t \log J^uT$, $0 < t < t_*$, for some determined constant $t_* > 1$. Here again, the construction relies on the study of a weighted transfer operators $\mathcal{L}_t$ acting on anisotropic Banach spaces of distributions. In this case, for each $t$, $\mathcal{L}_t$ satisfies a Lasota–Yorke type inequality, hence $\mathcal{L}_t$ is quasi-compact. Baladi and Demers then prove that $\mathcal{L}_t$ has a spectral gap, which in particular implies that $\mu_t$ has exponential mixing.

In this thesis (Chapter 3), we construct equilibrium states associated to piecewise Hölder potentials $g$ satisfying some additional assumptions. To do so, we use the same spaces $\mathcal{B}$ and $\mathcal{B}_w$ as in [BD20]. The transfer operator used is defined first on $C^1$ functions by

$$\mathcal{L}_g(f) := \left(e^g \frac{f}{J^sT}\right) \circ T^{-1}.$$

We also introduce a definition of topological pressure $P_*(T,g)$ which coincides with the one formulated by Bowen. In the case $g=0$, this quantity coincides with $h_*$ used in [BD20].

Recalling $\Lambda = 1 + 2\kappa_{\min}\tau_{\min} > 1$ the minimal expansion factor of $T$, we can state our first result on the Sinai billiard flow as follows.

**Theorem 1.4.2.** *If $g$ is a piecewise Hölder potential such that $P_*(T,g) - \sup g > s_0 \log 2$ and $\log \Lambda > \sup g - \inf g$, then there exists a unique equilibrium measure $\mu_g$. Furthermore $\mu_g$ is Bernoulli, $T$-adapted and has full support.*
*When $h_* > s_0 \log 2$, there exists a neighbourhood of the zero potential satisfying the above assumptions.*

We also introduce two technical assumptions, SSP.1 and SSP.2 – which stand for small singular pressure – weaker than the condition $\log \Lambda > \sup g - \inf g$. When $\log \Lambda > \sup g - \inf g$ is replaced by SSP.1, then the measure $\mu_g$ is only $T$-invariant and $T$-adapted. When $\log \Lambda > \sup g - \inf g$ is replaced by SSP.2, then the conclusions of Theorem 1.4.2 hold.

Furthermore we prove that

**Theorem 1.4.3.** *a) If $g$ satisfies the conditions $P_*(T,g) - \sup g > s_0 \log 2$ and SSP.2, then in the coordinates of the suspension, the measure $\bar{\mu}_g = (\mu_g(\tau))^{-1} \mu_g \otimes \lambda$ is a flow invariant measure. Furthermore $\bar{\mu}_g$ is Bernoulli, flow-adapted and has full support.*
*b) The set of equilibrium measures of $T$ under the potential $-h_{\mathrm{top}}(\phi_1)\tau$ is in bijection with the set of measures of maximal entropy for $\phi_t$.*

Chapter 4 is dedicated to the proof of the existence of a measure of maximal entropy for the billiard flow. As claimed in Theorem 1.4.3b), this is equivalent to prove that $T$ admits equilibrium measures under the potential $-h_{\mathrm{top}}(\phi_1)\tau$. To do so, we rely on the fact that $t \mapsto P_*(T, -t\tau) + t\tau_{\min}$ is decreasing and that $P_*(T, -h_{\mathrm{top}}(\phi_1)\tau) \geqslant 0$. Therefore, assuming $h_{\mathrm{top}}(\phi_1)\tau_{\min} > s_0 \log 2$, we get that $P_*(T, -t\tau) + t\tau_{\min} > s_0 \log 2$ for all $0 \leqslant t \leqslant h_{\mathrm{top}}(\phi_1)$. Then, we bootstrap from Theorem 1.4.2 by considering the supremum $t_\infty$ of the $t'$ such that for all $0 \leqslant t \leqslant t'$, $-t\tau$ has SSP.2. Thanks to Theorem 1.4.2, $t_\infty > 0$. Finally, assuming $t_\infty < h_{\mathrm{top}}(\phi_1)$ leads to a contradiction: using the Hölder inequality, we are able to construct a $t_2 > t_\infty$ which contradicts the maximality of $t_\infty$.

In other words, we prove

**Theorem 1.4.4.** *If $h_{\mathrm{top}}(\phi_1)\tau_{\min} > s_0 \log 2$, then there exists a unique measure of maximal entropy for the billiard flow. Furthermore, it is Bernoulli, flow-adapted and has full support.*

# Chapter 2

## Logarithmic bounds for ergodic sums of certain flows on the torus: a short proof

**Abstract**

This chapter contains the results of [Car22a](published in QTDS). We give a short proof that the ergodic sums of $\mathcal{C}^1$ observables for a $\mathcal{C}^1$ flow on $\mathbb{T}^2$ admitting a closed transversal curve whose Poincaré map has constant type rotation number have growth deviating at most logarithmically from a linear one. For this, we relate the latter integral to the Birkhoff sum of a well-chosen observable on the circle and use the Denjoy-Koksma inequality. We also give an example of a nonminimal flow satisfying the above assumptions.

## 2.1 Introduction

Since the work of Furstenberg [Fur73], it is known that the classical horocycle flow of a compact surface of constant negative curvature is uniquely ergodic — it has only one invariant Borel probability measure. This flow is related to a hyperbolic one, namely the geodesic flow, in the sense that the horocycle orbits are the unstable manifolds for the geodesic flow.

Using Symbolic Dynamics arguments (resp. equicontinuity of some functions), Marcus [Mar75a] (resp. [Mar75b]) generalized this result to the flow generated by the orientable one-dimensional unstable foliation of a connected basic piece of an Axiom A diffeomorphism (resp. flow). Later, Bowen and Marcus [BM77] extended this result to the higher dimensional strong stable or strong unstable foliation of a basic set for an Axiom A diffeomorphism or flow.

In their pioneer work, Giulietti and Liverani [GL19] focused on the one-dimensional stable foliation of a $\mathcal{C}^r$ Anosov diffeomorphism $F$ of the two-torus, inducing a flow $h^t$ called the Giulietti–Liverani (stable horocycle) flow (of $F$). Giulietti and Liverani proved that

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this flow is uniquely ergodic, minimal and that it admits a closed transverse curve such that the rotation number of the first return map to this curve is of constant type. For more basic facts about this flow, see [Bal22, Appendix A].

For any continuous function $f : \mathbb{T}^2 \to \mathbb{C}$, any $T > 0$ and any $x \in \mathbb{T}^2$, define the horocycle integral $H_{x,T}(f) = \int_0^T f(h^t(x))\,\mathrm{d}t$. By unique ergodicity, we have for any such $x$ and $f$,

$$\lim_{T\to\infty} \frac{H_{x,T}(f)}{T} = \mu^s(f) := \int_{\mathbb{T}^2} f\,\mathrm{d}\mu^s,$$

where $\mu^s$ is the unique invariant probability measure of the flow $h^t$.

For large enough $r$, Giulietti and Liverani introduce a transfer operator for $F$ on some suitable Banach space. Using eigenvectors of the dual operator associated to eigenvalues with modulus larger than the essential spectral radius (Ruelle resonances), they give an asymptotic expansion of $H_{x,T}(f)$ [GL19, Theorem 2.8]. The dominant term is the term $T\mu^s(f)$, corresponding to the trivial resonance $\lambda_0 = e^{h_{top}}$, where $h_{top}$ is the topological entropy of $F$. This expansion also involves a negative power law error term. A simpler asymptotic expansion, in the case where all Ruelle resonances of the transfer operator have trivial Jordan blocks, can be found in [Bal22, Equation (1.2)].

In their recent works, V. Baladi [Bal22] and G. Forni [For22] independently proved that horocycle integrals (in the set-up from [GL19]) do not have deviations, in other words the expansion is limited to the linear term with a bounded remainder. Their proofs are quite different: V. Baladi proves the strong result that the map $F$ does not have non-trivial Ruelle resonance, while G. Forni uses the action of the (pseudo-)Anosov diffeomorphism on the first cohomology — in the more general setting of surfaces of genus $g \geqslant 1$ (non-trivial Ruelle resonances can appear only for $g \geqslant 2$).

In this chapter we give a new, much shorter, proof of the absence of deviations for horocycle integrals by considering a slightly more general setting: we no longer assume that the flow can be obtained from the stable foliation of an Anosov diffeomorphism. Instead, we only assume that the flow can be recovered from the suspension of a circle diffeomorphism whose rotation number is of constant type. In particular, these flows are uniquely ergodic. For clarity, we call "ergodic integral" for this type of flows the quantity defined as "horocycle integral" previously.

We give an elementary proof that the ergodic integral of a $\mathcal{C}^1$ observable along the trajectory of such a flow on the two-torus grows at most logarithmically if the observable has zero average with respect to the unique invariant measure of the flow. This is the content of our main theorem (Theorem 2.2.2, corresponding to Theorem 1.4.1).

When comparing this estimate to the asymptotic expansion given by Giulietti and Liverani [GL19, Theorem 2.8], this result gives a new proof of the absence of deviations for the horocycle integral.

Finally, we prove that the class of flows we consider here is strictly larger than the class of flows studied by Giulietti and Liverani by constructing a flow satisfying our assumptions but which is not minimal — in contrast to all flows in [GL19]. This is the content of Theorem 2.3.1.

## 2.2 Main result

Given a flow $h_t$ on the two-torus, we call *ergodic integral* of an observable $f : \mathbb{T}^2 \to \mathbb{R}$ at $x \in \mathbb{T}^2$ and $T > 0$ the quantity $H_{x,T}(f) \coloneqq \int_0^T f \circ h_t(x)\,\mathrm{d}t$.

Recall the following classical theorem — we give a short proof of this fact using results from [KH95] in order to introduce notations for our main result. In particular the theorem below gives a simple sufficient condition for a flow to be written as the suspension of a circle diffeomorphism.

**Theorem 2.2.1.** *If $h_t$ is a $\mathcal{C}^1$ flow on the torus $\mathbb{T}^2$ without critical points nor periodic orbits, then there exists a smooth closed curve $\gamma$ transverse to $h_t$ such that $h_t$ is smoothly conjugated to the suspension of the first return map $R : \gamma \to \gamma$.*
*Moreover, the flow $h_t$ is uniquely ergodic, with a unique invariant measure $\mu$.*

Recall that an irrational number is of constant type if the sequence $(a_k)_k$ of its coefficients in its continued fraction expansion is bounded. We can now state our main result, using notations from the previous theorem.

**Theorem 2.2.2.** *If $h_t$ is a $\mathcal{C}^1$ flow on the torus $\mathbb{T}^2$ without critical point nor periodic orbit, and if the rotation number of the Poincaré first return map $R$ is of constant type, then there exist constants $K_1$ and $K_2$ such that for any $\mathcal{C}^1$ observable $f$ with $\int f\,\mathrm{d}\mu = 0$, any $x$ and any $T > 0$,*

$$|H_{x,T}(f)| \leqslant K_1 ||f||_{\mathcal{C}^1} \log(1+T) + K_2 ||f||_{\mathcal{C}^1}.$$

More precise versions of that estimate in the case of Giulietti–Liverani flows can be found in [Bal22] and in [For22]. The bound obtained by V.Baladi [Bal22] is much tighter — but the proof is longer — while the estimate given by G.Forni [For22] applies to flows on higher genus surfaces.

*Proof of Theorem 2.2.1.* By the Birkhoff recurrence theorem, any continuous transformation of a compact space has a recurrent point. Hence $h_1$ has recurrent orbits. In particular the flow $h_t$ also has recurrent points. By our assumptions on the flow, these orbits cannot be periodic. Hence, by [KH95, Propositions 14.2.1 and 14.2.2] there exists a smooth closed curve $\gamma$ *transverse* to $h_t$ and parametrised by $\mathbb{S}^1$ such that every orbit of $h_t$ intersects $\gamma$. We can therefore apply [KH95, Corollary 14.2.3] to get that $h_t$ is smoothly conjugated to the suspension flow of the first return map $R$ to $\gamma$. The conjugation is $\mathcal{C}^1$, since the change of coordinates is $(\theta, t) \mapsto h_t(\theta)$.

The map $R : \mathbb{S}^1 \to \mathbb{S}^1$ is a $\mathcal{C}^1$ diffeomorphism of the circle which has no periodic point. It is a classical result — see [CFS82, Theorem 3.3.5] — that $R$ is uniquely ergodic, with invariant measure $\nu$, and that its rotation number is irrational. From this, we deduce that $h_t$ is uniquely ergodic, with a unique invariant measure $\mu$. □

We can now give the proof of our main result.

*Proof of Theorem 2.2.2.* Suppose that the rotation number $\omega$ of $R$ is of constant type. In order to prove the estimate, we will compare the ergodic integral to the Birkhoff sum of an appropriate function.

Let $u : \mathbb{S}^1 \to \mathbb{R}_+$ be the first return time function to $\gamma$, and let $f : \mathbb{T}^2 \to \mathbb{R}$ be a $\mathcal{C}^1$-observable such that $\int_{\mathbb{T}^2} f\,\mathrm{d}\mu = 0$. By construction, $\gamma$ is a smooth curve, uniformly transverse to the flow, hence the function $u$ is of class $\mathcal{C}^1$. Define the $\mathcal{C}^1$ observable $g$ on $\gamma$ by the formula

$$g(x) = \int_0^{u(x)} f \circ h_t(x) \, \mathrm{d}t.$$

To estimate the ergodic integral of $f$ by the Birkhoff sum of $g$ under the map $R$, we use the following lemma.

**Lemma 2.2.3.** *For all $x \in \gamma$ and $T > 0$ there exists $n$ satisfying $\frac{T}{\sup(u)} - 1 \leqslant n \leqslant \frac{T}{\inf(u)}$ and such that*

$$\left| H_{x,T}(f) - \sum_{k=0}^{n-1} g \circ R^k(x) \right| \leqslant \sup(u) \sup |f|.$$

*For all $y \in \mathbb{T}^2$ there is $0 \leqslant \tau < \sup u$ and $x \in \gamma$ such that $y = h_\tau(x)$ and*

$$|H_{x,T+\tau}(f) - H_{y,T}(f)| \leqslant \sup(u) \sup |f|.$$

*Proof.* We first determine $n$. Since $\inf u > 0$, there exists $n$ such that $\sum\limits_{k=0}^{n-1} u \circ R^k(x) \leqslant T < \sum\limits_{k=0}^{n} u \circ R^k(x)$. Hence $n \inf u \leqslant T$ and $(n+1) \sup u \geqslant T$. Both estimates on ergodic integrals then follow from the fact that $h_t(R^n(x)) = h_{t + \sum_{k=0}^{n-1} u(R^k(x))}(x)$ for all $x \in \gamma$ and all $t \in \mathbb{R}$. □

In order to conclude by applying the Denjoy–Koksma theorem [Her79, Theorem VI.3.1], we also need the following lemma.

**Lemma 2.2.4.** *If $\omega = [0, a_1, \ldots, a_k, \ldots]$ is of constant type, then for any integer $n > 1$ there exists integers $N$ and $(n_1, \ldots, n_N)$ such that $n - 1 = \sum\limits_{k=0}^{N} n_k q_k$, where $\frac{p_k}{q_k} = [0, a_1, \ldots, a_k]$. Furthermore, we can choose $N < 4\log(n)/\log(2)$ and $n_k \leqslant B + 1$ for all $k$, where $B$ is a bound on the coefficients $(a_k)_{k \geqslant 1}$.*

*Proof.* Since the sequence $(q_k)_{k \geqslant 0}$ satisfies the recursion formula $q_{k+1} = a_k q_k + q_{k-1}$ with $q_0 = 1$ and $q_1 = a_1$, we get by induction that $2^{\frac{k-1}{2}} \leqslant q_k$. Therefore, there exists $N$ such that $q_N \leqslant n - 1 < q_{N+1}$ with the estimate $N < 4\log(n)/\log(2)$.

Define inductively the sequences $(r_k)_{0 \leqslant k \leqslant N+1}$ and $(n_k)_{0 \leqslant k \leqslant N}$ by $r_{N+1} := n - 1$ and the Euclidean division $r_{k+1} = n_k q_k + r_k$, with $0 \leqslant r_k < q_k$. Clearly, we get that $n - 1 = \sum\limits_{k=0}^{N} n_k q_k$ (because $q_0 = 1$). By contradiction, suppose there exists $k$ such that $n_k > B + 1$. Then

$$r_{k+1} = n_k q_k + r_k > (B+1) q_k + r_k > a_{k+1} q_k + q_{k-1} + r_k = q_{k+1} + r_k.$$

Therefore $r_{k+1} \geqslant q_{k+1}$, which is a contradiction. Hence $n_k \leqslant B + 1$ for all $k$. □

For completeness, we state the Denjoy–Koksma inequality:

**Theorem 2.2.5** (Denjoy–Koksma inequality)**.** *Let $f$ be a homeomorphism of the circle with an irrational rotation number $\rho(f)$. Let $\mu$ be a measure invariant by $f$, and let $p/q$ be such that $\gcd(p, q) = 1$ and $|q\rho(f) - p| < 1/q$. Then for all potentials $\varphi$ of bounded variation and all $x \in \mathbb{S}^1$, $\left| \sum\limits_{k=0}^{q-1} \varphi \circ f^k(x) - q \int \varphi \, \mathrm{d}\mu \right| < \mathrm{Var}(\varphi)$.*

Since $g$ is $\mathcal{C}^1$, it is of bounded variation. In addition, the denominators $(q_k)_{k\geqslant 0}$ associated to $\omega$ satisfy the assumption $|q_k\omega - p_k| < 1/q_k$ for some integer $p_k$ coprime with $q_k$. We can therefore apply the Denjoy–Koksma theorem to $g$, $R$ and any $q_k$. Furthermore notice that, by construction, $g$ is of $\nu$-average 0: indeed, let $M = \{(x,t) \mid x \in \gamma,\, t \in [0, u(x)]\}/\sim$, with $(x, u(x)) \sim (R(x), 0)$, be the space such that $h_t$ is conjugated with its unit speed vertical flow. Let $\bar{\mu}$ be the image of $\mu$ by the conjugacy map. Thus, $\bar{\mu}$ is invariant by the vertical flow and so it must be of the form $\bar{\mu} = \frac{1}{\int u \,\mathrm{d}\bar{\nu}} \bar{\nu} \otimes \mathrm{d}t$, where $\bar{\nu}$ is invariant under $R$. By unique ergodicity of $R$, we have $\bar{\nu} = \nu$. Thus

$$\begin{aligned} 0 &= \int_{\mathbb{T}^2} f \,\mathrm{d}\mu = \int_M f(h_t(x)) \,\mathrm{d}\bar{\mu}(x,t) \\ &= \frac{1}{\int u \,\mathrm{d}\nu} \int_\gamma \int_0^{u(x)} f(h_t(x)) \,\mathrm{d}t \,\mathrm{d}\nu(x) = \frac{1}{\int u \,\mathrm{d}\nu} \int g \,\mathrm{d}\nu \,. \end{aligned}$$

Fix $x \in \mathbb{T}^2$ and $T > 0$. By Lemma 2.2.3, there exist a point $y \in \gamma$ and an integer $n$ from which we can estimate the ergodic integral of $f$ at $x$ and $T$ with the Birkhoff sum of $R$ at $y$. In order to assume that $n > 1$, we assume that $T > 2 \sup u$ (otherwise, the theorem holds with $K_1 = 0$ and some $K_2 > 0$ depending only on $u$). By Lemma 2.2.4 we can decompose $n - 1$ as a sum from which we deduce the equality

$$\sum_{k=0}^{n-1} g \circ R^k(y) = \sum_{l=0}^{N} \sum_{m=0}^{n_l - 1} \sum_{k=0}^{q_l - 1} g \circ R^k \left( R^{m q_l + \sum_{i=0}^{l-1} n_i q_i} y \right).$$

From the Denjoy-Koksma inequality, for all $0 \leqslant l \leqslant N$, all $0 \leqslant m < n_l$ and all $y$ in $\gamma$,

$$\left| \sum_{k=0}^{q_l - 1} g \circ R^k \left( R^{m q_l + \sum_{i=0}^{l-1} n_i q_i} y \right) \right| < \mathrm{Var}(g),$$

we deduce the estimate

$$\left| \sum_{k=0}^{n-1} g \circ R^k(y) \right| \leqslant N(B+1)\,\mathrm{Var}(g) \leqslant \frac{4(B+1)\,\mathrm{Var}(g)}{\log 2} \log n \leqslant \frac{4(B+1)\,\mathrm{Var}(g)}{\log 2} \log \frac{T}{\inf(u)}.$$

Hence the result,

$$\begin{aligned} |H_{x,T}(f)| &\leqslant |H_{x,T}(f) - H_{y,T-\tau}(f)| + \left| H_{y,T-\tau}(f) - \sum_{k=0}^{n-1} g \circ R^k(y) \right| + \left| \sum_{k=0}^{n-1} g \circ R^k(y) \right|, \\ &\leqslant \frac{4B\,\mathrm{Var}(g)}{\log 2} \log \frac{T}{\inf(u)} + 2 \sup(u) \sup |f| =: \tilde{K}_1 \log T + \tilde{K}_2. \end{aligned}$$

We can bound the total variation $\mathrm{Var}(g)$ by the product of the length of $\gamma$ with $||g'||_{\mathcal{C}^0(\gamma)}$. By the definition of $g$, we get

$$||g'||_{\mathcal{C}^0(\gamma)} \leqslant ||u'||_{\mathcal{C}^0(\gamma)}\, ||f||_{\mathcal{C}^0} + ||u||_{\mathcal{C}^0(\gamma)} ||\mathrm{d}f||_{\mathcal{C}^0} \sup_{0 \leqslant t \leqslant ||u||_{\mathcal{C}^0(\gamma)}} ||\mathrm{d}h_t||_{\mathcal{C}^0}.$$

Notice that $||u'||_{\mathcal{C}^0(\gamma)}$ and $\sup_{0 \leqslant t \leqslant ||u||_{\mathcal{C}^0(\gamma}} ||\mathrm{d}h_t||_{\mathcal{C}^0}$ only depend on the flow $h_t$ and on $\gamma$. Hence there exist constants $K_1$ and $K_2$ that depend only on $h_t$ such that $\tilde{K}_1 \leqslant K_1 ||f||_{\mathcal{C}^1}$ and $\tilde{K}_2 \leqslant K_2 ||f||_{\mathcal{C}^1}$. □

Finally, remark that in order to get a rotation number of constant type, the condition for the flow not to have periodic orbit is necessary: otherwise the existence of a transverse curve $\gamma$ is no longer guaranteed. If such a curve exists then the first return map $R$ has a periodic point, hence has a rational rotation number.

## 2.3 A nonminimal flow satisfying the assumptions of Theorem 2.2.2

We finish this chapter by proving that the class of flows we are working with is strictly larger than the class of flows studied by Giulietti and Liverani which are necessarily minimal. The proof relies on constructing a family of $\mathcal{C}^1$ nonminimal flows. By [KH95, Proposition 14.2.4], these flows are less than $\mathcal{C}^2$.

**Theorem 2.3.1.** *There exists a flow on $\mathbb{T}^2$ satisfying the assumptions of Theorem 2.2.2 that is not minimal. Furthermore, the flow can be chosen to be renormalized by an Axiom A diffeomorphism.*

Notice however that all flows satisfying the assumptions of Theorem 2.2.2 are obtained by suspending circle diffeomorphisms of irrational rotation numbers, and thus are minimal on the support of their unique invariant measure.

Without the last condition of renormalization, we can simply construct such a flow by taking the suspension of a Denjoy counter-example whose rotation number is of constant type. Such circle diffeomorphisms exist by the original construction of Denjoy, which works for any irrational rotation number. For an expository on the construction of Denjoy counter-examples, see for example[1] [Ath15]. However, there is no reason for the flow obtained by suspending a Denjoy counter-example to be renormalized by an Axiom A diffeomorphism. Adding this condition, the flow falls into the category of $W^u$-flows studied by Marcus in [Mar75a], in the particular case where the phase space of the flow is the same as the one of the Axiom A map — as opposed to just the set of nonwandering points of the map. Finally, results on Ruelle spectrum and dynamical determinants for Axiom A diffeomorphisms can be found in [BT08, DR21] (and results on dynamical zeta functions for Axiom A flows in [DG18]), but asymptotic expansions of ergodic integrals associated to $W^u$-flows using transfer operator techniques are still quite rare in literature and there is room for work to be done in this setting.

In order to build a flow satisfying this last condition, consider the derived from Anosov transformation on the two-torus studied in [Cou16, Chapter 9] and [Cou06]. The flow $h_t$ will be the flow generated by some vector field $v^s_{\beta_0}$ defined below. Recall some notation. Starting from Arnold's *cat map* (case $\beta = 0$) in the diagonalized form, and adding a bump in the unstable direction, let $f_\beta : \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \to \mathbb{R}^2$ be as follows

$$f_\beta \begin{pmatrix} x \\ y \end{pmatrix} := \frac{1}{1+\lambda^2} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^2 + \beta k\left(\frac{\sqrt{x^2+y^2}}{2}\right) & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\lambda = \frac{1+\sqrt{5}}{2}$, $-\lambda^2 < \beta < 0$ and $k$ is an even, unimodal function supported in $[-1, 1]$ such that $k(0) = 1$ – *e.g.* $k(r) = (1-r^2)^2 \mathbb{1}_{[-1,1]}(r)$ – so that the map $f_\beta$ is invariant by the action

1. I thank Selim Ghazouani for suggesting this reference to me.

of $\mathbb{Z}^2$ and induces a map, also called $f_\beta$, on the torus $\mathbb{T}^2$. It is shown in [Cou16, Chapter 9] that $f_\beta$ is a diffeomorphism of class $C^1$ of the torus and if $-\lambda^2 < \beta < -\lambda^2 + 1$ then the origin is an attractive hyperbolic fixed point. Let $K_\beta$ be the invariant subset defined as the complement of the basin of attraction of 0. This map is an explicit example of Smale's derived from Anosov transformation as introduced in [Sma67, Section I.9], here obtained by perturbing Arnold's *cat map*.

Let $e_u = \frac{1}{\sqrt{1+\lambda^2}}\begin{pmatrix}\lambda\\1\end{pmatrix}$ and $e_s = \frac{1}{\sqrt{1+\lambda^2}}\begin{pmatrix}-1\\\lambda\end{pmatrix}$ be unitary eigenvectors of the matrix $A := \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$ respectively associated to eigenvalues $\lambda^2$ and $\lambda^{-2}$. Since $A$ is symmetric, notice that $(e_u, e_s)$ is an orthonormal basis. In this basis the Jacobian matrix of $f_\beta$ at $x \in \mathbb{T}^2$ is

$$\operatorname{Jac}(f_\beta)(x) = \begin{pmatrix}a_\beta(x) & b_\beta(x)\\0 & \lambda^{-2}\end{pmatrix}.$$

Since the Jacobian is upper-triangular, lines spanned by $e_u$ are stable by $f_\beta$. Assuming that $k$ satisfies also $k + \mathrm{id}_{\mathbb{R}}\, k' \leqslant 1$, $f_\beta|_{K_\beta}$ expands uniformly the direction spanned by $e_u$. In order to construct a stable foliation over $K_\beta$, for $X$ a vector field, denote $(f_\beta)_* X(x) = (\mathrm{d}_x f_\beta)^{-1} X(f_\beta(x))$ to be the pullback of $X$ by $f_\beta$. Formally, if $v^s_\beta = \lim_{n\to+\infty} \lambda^{-2n}(f_\beta)^n_* X$, then $\lambda^{-2}(f_\beta)_* v^s_\beta = v^s_\beta$, or in other words $\mathrm{d}_x f_\beta\, v^s_\beta(x) = \lambda^{-2} v^s_\beta(f(x))$, $v^s_\beta$ is uniformly contracted by $\mathrm{d}f_\beta$. For the constant vector field $X \equiv e_s$, formally we get

$$v^s_\beta(x) = e_s - \sum_{i=0}^{\infty} \lambda^{-2i} b_\beta(f^i_\beta(x)) \prod_{j=0}^{i} \frac{1}{a_\beta(f^j_\beta(x))}\, e_u, \quad x \in \mathbb{T}^2. \tag{2.3.1}$$

This equation being only formal, we need to check that the series inside it converges. Since $b_\beta$ is bounded and $a_\beta > 1$ on the compact set $K_\beta$, (2.3.1) defines a vector field on $K_\beta$, uniformly contracted by $f_\beta$:

$$\mathrm{d}_x f_\beta\, v^s_\beta(x) = \lambda^{-2} v^s_\beta(f_\beta(x)) \tag{2.3.2}$$

for all $x \in K_\beta$. It is shown in [Car22b, Theorems 3.3 and 3.6] — in a slightly more general context — that (2.3.1) defines a Lipschitz continuous vector field on $\mathbb{T}^2$ for any fixed $\beta$ in $]-\lambda^2+\lambda^{-4}, 0]$ and that the map $(x,\beta) \mapsto v^s_\beta(x)$ is continuous on $\mathbb{T}^2\times\,]-\lambda^2+\lambda^{-4}, 0]$.

Let $h_t$ be the flow generated by $v^s_{\beta_0}$ for some fixed $-\lambda^2+\lambda^{-4} < \beta_0 < -\lambda^2+1$. In fact, if we choose for the function $k$ any $\mathcal{C}^2$ unimodal and even function supported in $[-1,1]$, equal to 1 at 0 and satisfying $k + id\, k' \leqslant 1$, the induced vector fields $v^s_\beta$ enjoys the same properties as before, but they are also $\mathcal{C}^1$ – see the discussion in [Car22b, Theorem 3.7] – hence the flow $h_t$ is also $\mathcal{C}^1$. We make such a choice for $k$. We claim that this flow $h_t$ satisfies the condition of Theorem 2.2.2 and that it is not minimal.

In order to prove this result, we first construct a closed transversal curve $\gamma$. We then construct a particular homotopy between the first return map and a rigid rotation, where none of the in-between maps has a periodic point. From the continuity of the rotation number, it is enough to compute the rotation number of the rigid rotation, which happens to be a quadratic integer. The nonminimality follows from the invariance of the proper closed set $K_{\beta_0}$ by the flow $h_t$. First we need the following lemma.

**Lemma 2.3.2.** *The flow $h_t$ does not have periodic orbit. This is also true for the flow generated by $v^s_\beta$ for any $-\lambda^2+\lambda^{-4}<\beta\leqslant 0$.*

*Proof.* By construction, each vector field $v^s_\beta$ satisfies $\mathrm{d}_x f_\beta(v^s_\beta(x)) = \lambda^{-2}v^s_\beta(f_\beta(x))$. By differentiating $f_{\beta_0}\circ h_t(x)$ and $h_{\lambda^{-2}t}\circ f_{\beta_0}(x)$ according to $t$, we get that these two functions satisfy the same Cauchy problem for all $x\in\mathbb{T}^2$, thus the relation

$$f_{\beta_0}\circ h_t = h_{\lambda^{-2}t}\circ f_{\beta_0} \tag{2.3.3}$$

holds by uniqueness of the solution (because $v^s_\beta$ is Lipschitz continuous). Therefore, if by contradiction $h_t$ has a periodic orbit, by applying $f^n_{\beta_0}$, for $n$ large enough, we get an arbitrarily short periodic orbit for the flow. This contradicts the fact that the component along $e_s$ in the basis $(e_u,e_s)$ of $v^s_{\beta_0}$ is constant equal to 1. □

*Proof of Theorem 2.3.1.* Since the map $(x,\beta)\mapsto v^s_\beta(x)$ is continuous on the compact set $\mathbb{T}^2\times[\beta_0,0]$, the component of these vector fields in the basis $(e_u,e_s)$ along $e_u$ is uniformly bounded and along $e_s$ is equal to 1, by definition. Therefore, there exists a vector $w$ of rational slope, say $w=\frac{1}{\sqrt{p^2+q^2}}\begin{pmatrix}q\\p\end{pmatrix}$, where $p$ and $q$ are coprimes, so that $w$ is uniformly transverse to $v^s_\beta$ for all $\beta\in[\beta_0,0]$. Define $\gamma$ to be the closed curve passing through $(0,0)$ and with slope $p/q$. By choice of $w$, the curve $\gamma$ is transverse to $v^s_\beta$ and so for every $\beta$ in $[\beta_0,0]$. We can naturally parametrize $\gamma$ by $\mathbb{S}^1$.

Let $R:\mathbb{S}^1\to\mathbb{S}^1$ be the first return map to $\gamma$ of $h_t$. Notice that performing a time change on this flow does not affect the first return map $R$, but only the first return time function $u$. In order to simplify computations, renormalize the vector fields as follows

$$w^s_\beta = \frac{1}{\left\langle v^s_\beta, w^\perp\right\rangle} v^s_\beta$$

so that, for each $\beta$, the flow $\phi^{(\beta)}_t$ generated by $w^s_\beta$ has a constant first return time function $u_\beta\equiv\tau_\beta$, where $w^\perp$ is the unitary vector equal to $w$ rotated by an angle $\pi/2$. These first return time functions do not depend on $\beta$, in other words $\tau_\beta\equiv\tau$. Since $b_0\equiv 0$, notice that $w^s_0$ is a constant vector field (equals everywhere to $e_s$), hence its first return map to $\gamma$ is a rigid translation $R_\alpha: x\mapsto x+\alpha$. Introduce also the notation $R^{(\beta)}$ for the first return map to $\gamma$ of $\phi^{(\beta)}_t$. In particular $R=R^{(\beta_0)}$ and $R_\alpha=R^{(0)}$.

By [Car22b, Theorem 3.10], the map $\beta\mapsto v^s_\beta$ is continuous for the $\mathcal{C}^0$-topology on the space of vector fields. From a Gronwall type argument, we get that $\beta\mapsto R^{(\beta)}$ is continuous for the $\mathcal{C}^0$-topology. Now, by [Her79, Proposition II.2.7], the map $\beta\mapsto\rho(R^{(\beta)})$ is continuous, where $\rho(R^{(\beta)})$ stands for the rotation number of $R^{(\beta)}$. In order to prove that $\rho(R)=\alpha$, we prove that $\rho(R^{(\beta)})$ cannot be rational, but this directly follows from Lemma 2.3.2. Hence $\beta\mapsto\rho(R^{(\beta)})$ is a constant map and $\rho(R)=\alpha$.

We now compute the value of $\alpha$. Consider lifts $\tilde{w}^s_{(0)}$, $\tilde{\gamma}$ and $\tilde{\phi}^{(0)}_t$ to $\mathbb{R}^2$ of respectively $w^s_0$, $\gamma$ and $\phi^{(0)}_t$. Let $(\partial_x,\partial_y)$ be the canonical basis of $\mathbb{R}^2$. Notice that the arc $\{\tilde{\phi}^{(0)}_t((0,1)) \mid -p\tau\leqslant t\leqslant 0\}$ starts at the point $(0,1)$ and ends on the branch of $\tilde{\gamma}$ containing $(0,0)$ at some point $cw$, for some $c>0$. The coordinates of this intersection point satisfy the system of equations

$$\begin{cases} -p\tau\left\langle w^s_{(0)},\partial_x\right\rangle = cq(p^2+q^2)^{-1/2}\\ 1-p\tau\left\langle w^s_{(0)},\partial_y\right\rangle = cp(p^2+q^2)^{-1/2}, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product. Now, notice that $c/|\gamma| = -p\alpha$, where $|\gamma|$ is the length of the closed curve $\gamma$. We can solve these equations for $\alpha$ and get

$$\alpha = \frac{1}{pq} \frac{1}{\lambda - \frac{p}{q}}$$

which clearly is a quadratic integer, since $\lambda$ is. Therefore $\alpha$ is of constant type.

The nonminimality of $h_t$ is ensured by properties proven in [Cou16, Chapter 9]. More precisely, let $U$ be the basin of attraction of $(0,0)$ for $f_{\beta_0}$ and $K$ be its complement in the torus. In [Cou16, Chapter 9], Coudène proved that the set $K$ is nonempty and that $U$ and $K$ are invariant by $f_{\beta_0}$. Now, because of (2.3.3), the sets $U$ and $K$ are invariant by the flow $h_t$.

Finally, the map $f$ is an Axiom A diffeomorphism since $f$ is transitive [Cou16, Chapter 9] on the hyperbolic set $K$ [Car22b, Theorem 2.9]. Therefore, by the shadowing lemma, periodic points are dense in the compact invariant set $K$ which coincides with the nonwandering set of $f$. □

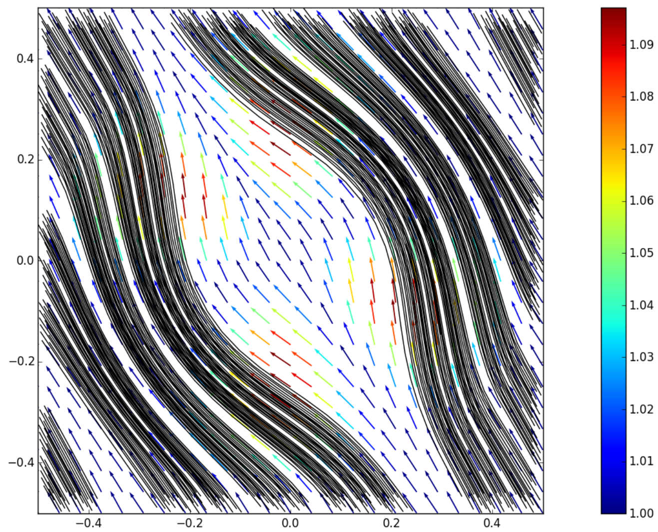


**Figure 2.1 –** Representation of the minimal component $K$ of the flow $(h_t)$. Underneath is the vector field $v^s$ generating the flow.

Finally, we give in Figure 2.1 a representation of the set $K$. In [Cou16, Chapter 9], it is proven that $K$ is the closure of the stable leaf $W^s(p)$ of a hyperbolic fixed point $p$ for $f_{\beta_0}$. From the relation (2.3.3) and the Hartman-Grobman theorem, it follows that this stable leaf is equal to the orbit of $p$ by the flow $h_t$. From [CFS82, Theorem 3.3.4], the set $K \cap \gamma$ coincides with any $\omega$-limit set and any $\alpha$-limit set of $R$. Therefore, the set $K$ is the minimal component of $h_t$ and is also an attractor for both positive and negative times. Moreover, $K$ is also the support of the unique invariant measure $\mu$ of $h_t$.

## 2.A Alternative proof of Theorem 2.3.1 from semi-conjugacy

We give an alternative proof of Theorem 2.3.1. More precisely, we use the same example, but we compute the rotation number in a different way: we construct a semi-conjugacy map $h$ so that $h \circ R = R_\alpha \circ h$. It will follow that the rotation number of $R$ is $\alpha$. The construction of $h$ is inspired from the proof of [Yoc05, Proposition 7].

*Proof.* Exactly as in the first proof of Theorem 2.3.1, we construct the closed transversal curve $\gamma$ and we renormalize the vector fields $v^s_\beta$ so that the time of first return function to $\gamma$ of their associated flows is constant. The computation of $\alpha$ remains the same, and we get that $\alpha$ is a quadratic integer, hence $\alpha$ is of constant type. In particular, the rotation $R_\alpha$ is minimal.

We now prove that the first return map $R$ of $h_t$ is semi-conjugated to $R_\alpha$. To this end, we construct a surjective and continuous function $h$ of the circle.

Let $h(R^n(0)) \coloneqq R^n_\alpha(0)$ for all $n \in \mathbb{Z}$. This map is well defined since $h_t$ has no periodic orbit by Lemma 2.3.2, so neither does $R$. In order to extend $h$ into a continuous map, we first prove that it preserves order of triplets. Fix an orientation of $\mathbb{S}^1$ — and therefore of $\gamma$ — seen as $\mathbb{R}/\mathbb{Z}$. Let $x_1 \coloneqq R^{n_1}(0)$, $x_2 \coloneqq R^{n_2}(0)$ and $x_3 \coloneqq R^{n_3}(0)$ be so that $(x_1, x_2, x_3)$ is an ordered triplet of $\mathbb{S}^1$ — we can assume that $n_1$, $n_2$ and $n_3$ are distinct. We prove that the triplet $(x'_1, x'_2, x'_3) = (h(x_1), h(x_2), h(x_3))$ is also ordered. Consider the family of curves $\varphi_\beta \coloneqq \{\phi^{(\beta)}_t(0) \mid \min(n_1, n_2, n_3)\tau \leqslant t \leqslant \max(n_1, n_2, n_3)\tau\}$. By continuity of $(x, \beta) \mapsto w^s_\beta(x)$, this family depends on $\beta$ in a continuous fashion.

Notice that points $x_1$, $x_2$ and $x_3$ correspond to some intersection points between $\varphi_{\beta_0}$ and $\gamma$, and that points $x'_1$, $x'_2$, and $x'_3$ correspond to some intersection points between $\varphi_0$ and $\gamma$. Furthermore, we can connect $x_1$ to $x'_1$ (respectively $x_2$ to $x'_2$, and $x_3$ to $x'_3$) with intersection points between $\gamma$ and $\varphi_\beta$ when varying the value of $\beta$. Therefore we can track the evolution of $(x_1, x_2, x_3)$ with continuous functions $(x_1(\beta), x_2(\beta), x_3(\beta))$ of $\beta$ such that $x_1(\beta_0) = x_1$ and $x_1(0) = x'_1$ — and similarly for $x_2(\beta)$ and $x_3(\beta)$.

By contradiction, suppose that the triplet $(x'_1, x'_2, x'_3)$ is not ordered. By continuity, this means that for some value of $\beta_1$ in $[\beta_0, 0]$ and without loss of generality $x_1(\beta_1) = x_2(\beta_1)$. In other words, this means that the first return map to $\gamma$ of $\phi^{(\beta_1)}_t$ has a periodic point, which contradicts Lemma 2.3.2.

Therefore, the map $h$ can be lifted into a "degree" one, increasing, function $\tilde{h} : \pi^{-1}\{R^n(0) \mid n \in \mathbb{Z}\} \to \pi^{-1}\{R^n_\alpha(0) \mid n \in \mathbb{Z}\}$, where $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the canonical projection. In other words, $\pi \circ \tilde{h} = h \circ \pi$ and $\tilde{h}(x+1) - \tilde{h}(x) = 1$ for all $x$ where $\tilde{h}$ is defined. By minimality of $R_\alpha$, the range of $\tilde{h}$ is dense in $\mathbb{R}$. Hence, we can uniquely extend $\tilde{h}$ by a continuous, increasing and surjective function $\tilde{h} : \mathbb{R} \to \mathbb{R}$. Its projection on the circle, still noted $h$, is also continuous and extends $h$ into a degree one map of the circle. By continuity of $R$ and of $R_\alpha$, we get that $h \circ R = R_\alpha \circ h$. Therefore, by [Her79, Proposition II.2.10], the rotation number of $R$ is $\alpha$, a quadratic integer.

The nonminimality of $h_t$ is ensured by properties proven in [Cou16, Chapter 9]. □

*Remark* 2.A.1. The construction of the conjugacy map $h$ comes from the following heuristic. Since the stable manifold of 0 under the cat map is blown up into an open set, the basin of attraction $U_\beta \coloneqq \mathbb{T}^2 \setminus K_\beta$ of 0 under $f_\beta$, we expect that the map $h$ relates the orbit of 0 under $R_\alpha$ with the orbit of $I$ under $R$, where $I$ is the connected component of $\gamma \cap U_\beta$ containing 0 (notice that $I$ is a wandering interval and that its orbit under $R$ is $\gamma \cap U_\beta$, which is dense in $\gamma$). More precisely, we expect $h$ to be similar to the Cantor staircase function, being constant when restricted to each $R^n(I)$. As in the construction of the Cantor staircase function, we only need to know the values of $h$ where it is constant, as long as $h$ is non-decreasing and that this set of values has a connected closure. In the proof above, we chose to define $h$ first by setting $h(x_n) = R_\alpha^n(0)$ with $x_n = R^n(0)$, but we could have chosen any sequence $x_n \in R^n(I)$.

# Chapter 3

## A family of natural equilibrium measures for Sinai billiard flows

**Abstract**

This chapter contains the results of [Car22c]. The Sinai billiard flow on the two-torus, i.e., the periodic Lorentz gas, is a continuous flow, but it is not everywhere differentiable. Assuming finite horizon, we relate the equilibrium states of the flow with those of the Sinai billiard map $T$ – which is a discontinuous map. We propose a definition for the topological pressure $P_*(T, g)$ associated to a potential $g$. We prove that for any piecewise Hölder potential $g$ satisfying a mild assumption, $P_*(T, g)$ is equal to the definitions of Bowen using spanning or separating sets. We give sufficient conditions under which a potential gives rise to equilibrium states for the Sinai billiard map. We prove that in this case the equilibrium state $\mu_g$ is unique, Bernoulli, adapted and gives positive measure to all nonempty open sets. For this, we make use of a well chosen transfer operator acting on anisotropic Banach spaces, and construct the measure by pairing its maximal eigenvectors. Last, we prove that the flow invariant probability measure $\bar{\mu}_g$, obtained by taking the product of $\mu_g$ with the Lebesgue measure along orbits, is Bernoulli and flow adapted. We give examples of billiard tables for which there exists an open set of potentials satisfying those sufficient conditions.

## 3.1 Introduction

### 3.1.1 Billiards and equilibrium states

In this work, we are concerned with a class of dynamics with singularities: the dispersing billiards introduced by Sinai [Sin70] on the two-torus. A Sinai billiard on the torus is (the quotient modulo $\mathbb{Z}^2$, for position, of) the periodic planar Lorentz gas (1905) model for the motion of a single dilute electron in a metal. The scatterers (corresponding to atoms of

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the metals) are assumed to be strictly convex (but not necessarily discs). Such billiards have become fundamental models in mathematical physics.

To be more precise, a Sinai billiard table $Q$ on the two-torus $\mathbb{T}^2$ is a set $Q = \mathbb{T}^2 \setminus B$ with $B = \sqcup_{i=1}^{D} B_i$ for some finite number $D \geqslant 1$ of pairwise disjoint closed domains $B_i$, called scatterers, with $C^3$ boundaries having strictly positive curvature – in particular, the scatterers are strictly convex. The billiard flow $\phi_t$ is the motion of a point particle travelling at unit speed in $Q$ with specular reflections off the boundary of the scatterers. Identifying outgoing collisions with incoming ones in the phase space, the billiard flow is continuous. However, the grazing collisions – those tangential to scatterers – give rise to singularities in the derivative [CM06]. The Sinai billiard map $T$ – also called collision map – is the return map of the single point particle to the scatterers. Because of the grazing collisions, the Sinai billiard map is a discontinuous map.

Sinai billiard maps and flows both preserve smooth invariant probability measures, respectively $\mu_{\mathrm{SRB}}$ and $\tilde{\mu}_{\mathrm{SRB}}$, which have been extensively studied: $(T, \mu_{\mathrm{SRB}})$ and $(\phi_t, \tilde{\mu}_{\mathrm{SRB}})$ are uniformly hyperbolic, ergodic, K-mixing [Sin70,BS73,SC87], and Bernoulli [GO74,CH96]. The measure $\mu_{\mathrm{SRB}}$ is $T$-adapted [KSLP86] in the sense of the integrability condition:

$$\int |\log d(x, \mathcal{S}_{\pm 1})| \, \mathrm{d}\mu_{\mathrm{SRB}} < \infty ,$$

where $\mathcal{S}_{\pm 1}$ is the singularity set for $T^{\pm 1}$. Both systems enjoy exponential decay of correlations [You98, DZ11]. Since the billiard has many periodic orbits, it thus has many other ergodic invariant measures, but until very recently most of the results apply to perturbations of $\mu_{\mathrm{SRB}}$ [CWZ17, DRBZ18].

In the case of an Anosov flow, it is known since the work of Bowen [Bow72b] that the Kolmogorov-Sinai entropy is upper-semicontinuous, which guarantees the existence of measures of maximal entropy, or more generally, of equilibrium states. Because of the singularities, billiard flows are not Anosov and therefore methods used in the context of Anosov flows cannot be applied easily. The upper-semicontinuity of the entropy is not known at the moment, and, more generally, the existence of equilibrium states has to be treated one potential at the time.

In a recent paper, Baladi and Demers [BD20] proved, under a mild technical assumption and assuming finite horizon, that there exists a unique measure of maximal entropy $\mu_*$ for the billiard map, and that $\mu_*$ is Bernoulli, $T$-adapted, charges all nonempty open sets and does not have atoms. Their construction of this measure relies on the use of a transfer operator acting on anisotropic Banach spaces, similar to those used by [DZ11] in order to study $\mu_{\mathrm{SRB}}$. Combining their work with those of Lima–Matheus [LM18] and Buzzi [Buz20], Baladi and Demers proved that their exists a positive constant $C$ such that

$$Ce^{h_* m} \leqslant \# \mathrm{Fix}\, T^m , \quad \forall m \geqslant 1 , \tag{3.1.1}$$

where $\#\mathrm{Fix}\, T^m$ denotes the number of fixed points of $T^m$, and $h_*$ is the topological entropy of the map $T$ from [BD20]. Baladi and Demers also give a condition under which $\mu_*$ and $\mu_{\mathrm{SRB}}$ coincide.

In a subsequent paper, Baladi and Demers [BD22] constructed a family of equilibrium states $\mu_t$ for $T$ associated to the family of geometric potentials $-t \log J^u T$, where $J^u T$ is the unstable Jacobian of $T$ and $t \in (0, t_*)$ for some $t_* > 1$. In the case $t = 1$, $\mu_t = \mu_{\mathrm{SRB}}$. The construction again relies on the use of a family of transfer operators $\mathcal{L}_t$ acting on anisotropic

Banach spaces. For each $t \in (0, t_*)$, they proved that $\mu_t$ is the unique equilibrium state associated with the potential $-t \log J^u T$, that $\mu_t$ is mixing, $T$-adapted, has full support and does not have atoms. Baladi and Demers also showed that each transfer operator $\mathcal{L}_t$ has a spectral gap, from which they deduced the exponential rate of mixing for each measure $\mu_t$, for $C^1$ observables.

Even more recently, Demers and Korepanov [DK22] proved a polynomial decay of correlations for the measure $\mu_*$ for Hölder observables.

In this paper, we give a sufficient condition under which a piecewise Hölder potential $g$ admits equilibrium states for $T$. Under this assumption, we prove that the equilibrium state is in fact unique, Bernoulli, $T$-adapted and charges all nonempty open sets. We prove that its lift into a flow invariant measure is Bernoulli and flow-adapted. We also identify the potential $g = -h_{\text{top}}(\phi_1)\tau$ to be such that its corresponding equilibrium states for $T$ – whenever they exist – are in bijection with measures of maximal entropy of the billiard flow.

Notice that the geometric potentials $-t \log J^u T$ are not piecewise Hölder, and thus the work of Baladi and Demers [BD22] on those potentials is distinct from ours.

### 3.1.2 Statement of main results – Organization of the paper

Since transfer operator techniques are simpler to implement for maps than for flows, we will be concerned with the associated billiard map $T : M \to M$ defined to be the first collision map on the boundary of $Q$, where $M = \partial Q \times [-\pi/2, \pi/2]$. We assume as in [You98, BD20], that the billiard table $Q$ has *finite horizon*, in the sense that the billiard flow does not have any trajectories making only tangential collisions – in particular, this implies that the return time function $\tau$ to a scatterers is bounded.

The first step is to find a suitable notion of topological pressure $P_*(T, g)$ for the discontinuous map $T$ and a potential $g : M \to \mathbb{R}$. In order to define it, we introduce as in [BD20], the following increasing family of partition of $M$. Let $\mathcal{P}$ be the partition into maximal connected sets on which both $T$ and $T^{-1}$ are continuous, and let $\mathcal{P}_{-k}^n = \bigvee_{i=-k}^n T^{-i}\mathcal{P}$. Then the sequence $\sum_{P \in \mathcal{P}_0^n} \sup_P e^{S_n g}$ is submultiplicative, where $S_n g = \frac{1}{n}\sum_{i=0}^{n-1} g \circ T^i$ is the Birkhoff sum of $g$. We can thus define the topological pressure by

**Definition 3.1.1.** $P_*(T, g) := \lim\limits_{n \to +\infty} \frac{1}{n} \log \sum\limits_{P \in \mathcal{P}_0^n} \sup_P e^{S_n g}$

Section 3.2 is dedicated to the study of this quantity. In particular, we prove (Proposition 3.2.2) that whenever the potential $g$ is smooth enough – piecewise Hölder – and $P_*(T, g) - \sup g > 0$ then $P_*(T, g)$ coincides with both Bowen's definitions using spanning sets and separating sets. We also prove (Lemma 3.2.4) that for each $T$-invariant measure $\mu$, we have $P_*(T, g) \geqslant h_\mu(T) + \int g \,\mathrm{d}\mu$. Finally, we show that if $g = -h_{\text{top}}(\phi_1)\tau$ admits an equilibrium state $\mu_g$, then the measure $\bar{\mu}_g = (\int \tau \,\mathrm{d}\mu_g)^{-1}\mu_g \otimes \lambda$ is a measure of maximal entropy for the billiard flow, seen as a suspension flow over $T$, where $\lambda$ is the Lebesgue measure in the flow direction.

To state our existence results (in Section 3.6), we need to quantify the recurrence to the singular set. Fix an angle $\varphi_0$ close to $\pi/2$ and $n_0 \in \mathbb{N}$. We say that a collision is $\varphi_0$-grazing if its angle with the normal is larger than $\varphi_0$ in absolute value. Let $s_0 = s_0(\varphi_0, n_0) \in (0, 1]$

denote the smallest number such that

$$\text{any orbit of length } n_0 \text{ has at most } s_0 n_0 \text{ collisions which are } \varphi_0\text{-grazing.} \tag{3.1.2}$$

Due to the finite horizon condition, we can choose $\varphi_0$ and $n_0$ such that $s_0 < 1$. We refer to [BD20, §2.4] for further discussion on this quantity. From [CM06], $\Lambda = 1 + \kappa_{\min}\tau_{\min} > 1$ is the expanding factor in the hyperbolicity of $T$, where $\kappa_{\min}$ is the minimal curvature of the scatterers and $\tau_{\min}$ is the minimum of the return time function $\tau$. Define $\mathcal{S}_0 = \{(r, \varphi) \in M \mid |\varphi| = \pi/2\}$ the set of grazing collisions, and $\mathcal{S}_{\pm n} = \cup_{i=0}^{n} T^{\mp i}\mathcal{S}_0$ the singular set of $T^{\pm n}$. Call $\mathcal{N}_\varepsilon(\cdot)$ the $\varepsilon$-neighbourhood of a set. Then

**Theorem 3.1.2.** *If $g$ is a bounded, piecewise Hölder potential such that $P_*(T, g) - \sup g > s_0 \log 2$ and $\log \Lambda > \sup g - \inf g$, then there exists a probability measure $\mu_g$ such that*

*(i) $\mu_g$ is $T$-invariant, $T$-adapted and for all $k \in \mathbb{Z}$, there exists $C_k > 0$ such that $\mu_g(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \leqslant C_k |\log \varepsilon|^{-\gamma}$, where $\gamma > 1$ is such that $P_*(T, g) - \sup g > \gamma s_0 \log 2$.*

*(ii) $\mu_g$ the unique equilibrium state of $T$ under $g$: that is $P_*(T, g) = h_{\mu_g}(T) + \int g \, \mathrm{d}\mu_g$ and $P_*(T, g) > h_\mu(T) + \int g \, \mathrm{d}\mu$ for all $\mu \neq \mu_g$.*

*(iii) $\mu_g$ is Bernoulli[1] and charges all nonempty open sets.*

*If the assumption $\log \Lambda > \sup g - \inf g$ is weakened into the condition SSP.1 (as defined above Lemma 3.3.3), then item (i) still holds. If the assumption $\log \Lambda > \sup g - \inf g$ is weakened into the condition SSP.2 (as defined above Corollary 3.3.6), then items (i), (ii) and (iii) hold.*

The above theorem will follow from Proposition 3.6.1, Lemma 3.6.2, Corollary 3.6.14, and Propositions 3.6.18, 3.6.15, 3.6.10. Furthermore, assuming the sparse recurrence to singularities condition from [BD20], we provide in Remark 3.3.11 an open set of potentials, each having SSP.1 and SSP.2.

The tool used to construct the measure $\mu_g$ is a transfer operator $\mathcal{L}_g$ with $\mathcal{L}_g f = (f\, e^g / J^s T) \circ T^{-1}$, similar to the one used in [BD20] corresponding to the case $g \equiv 0$. This operator and the anisotropic Banach spaces on which it acts are defined in details in Section 3.4. Section 3.3 contains key combinatorial growth lemmas, controlling the growth in weighted complexity of the iterates of a stable curve. These lemmas will be crucial since the quantity they control appears in the norms of the iterates of $\mathcal{L}_g$. In Section 3.5, we prove a (degenerated) “Lasota–Yorke" type inequality (Proposition 3.5.1) – giving an upper bound on the spectral radius of $\mathcal{L}_g$ – as well as a lower bound on the spectral radius (Theorem 3.5.3).

Section 3.6 is devoted to the construction and the properties of the measure $\mu_g$. From the estimates on the norms from the previous section, we are able to construct left and right maximal eigenvectors ($\tilde{\nu}$ and $\nu$) for $\mathcal{L}_g$. We construct the measure $\mu_g$ by pairing these eigenvectors. We then prove the estimates on the measure of a neighbourhood of the singular sets (Lemma 3.6.2). Section 3.6.3 contains the key result of the absolute continuity of the stable and unstable foliations with respect to $\mu_g$, as well as the proof that $\mu_g$ has total support – this is done by extending $\nu$ into a measure and exploiting the $\nu$-almost everywhere positive length of unstable manifolds from Section 3.6.2. In

1. Recall that Bernoulli implies K-mixing, which implies strong mixing, which implies ergodic.

Section 3.6.4, we show that $\mu_g$ is ergodic, from which we bootstrap to K-mixing using a Hopf argument. Adapting [CH96] with modifications from [BD20], we deduce from the hyperbolicity and the K-mixing that $\mu_g$ is Bernoulli. Still in Section 3.6.4, we give an upper-bound on the measure of weighted Bowen balls, from which we deduce, using the Shannon–MacMillan–Breiman theorem, that $\mu_g$ is an equilibrium state for $T$ under the potential $g$ (Corollary 3.6.14). Finally, the Section 3.6.5 is dedicated to the uniqueness of the equilibrium state $\mu_g$.

In Section 3.7, we prove using arguments from [CM06] that $(\phi_t, \bar{\mu}_g)$ is K-mixing (Proposition 3.7.1), and again, using the hyperbolicity of the billiard flow, we adapt [CH96] in order to prove that $(\phi_t, \bar{\mu}_g)$ is Bernoulli (Proposition 3.7.2). Finally, we prove that $\bar{\mu}_g$ is flow adapted in the sense of the integrability condition formulated in Proposition 3.7.4. We summarize these results about the billiard flow in the following theorem.

**Theorem 3.1.3.** *Let $g$ be a potential satisfying the assumptions from Theorem 3.1.2, and let $\bar{\mu}_g := (\int \tau \, \mathrm{d}\mu_g)^{-1} \mu_g \otimes \lambda$. Then $\bar{\mu}_g$ is a $\phi_t$-invariant Borel probability measure that is an equilibrium states for $\phi_t$ and any potential $\tilde{g}$ such that $g = \lambda(\tilde{g}) - P(\phi_1, \tilde{g})\tau$, where $\lambda(\tilde{g})(x) = \int_0^{\tau(x)} \tilde{g}(\phi_t(x)) \, \mathrm{d}t$. Furthermore, $\bar{\mu}_g$ is flow adapted and $(\phi_t, \bar{\mu}_g)$ is Bernoulli.*

In a work in preparation with Baladi and Demers [BCD22], we bootstrap from the results of the present paper to show that if $h_{\mathrm{top}}(\phi_1)\tau_{\min} > s_0 \log 2$ then the potential $-h_{\mathrm{top}}(\phi_1)\tau$ satisfies the sufficient assumptions SSP.1 and SSP.2 in our Theorem 3.1.2, thus constructing a measure of maximal entropy for the billiard flow. This is done by studying the family of potentials $-t\tau$ and proving that the maximal value $t_\infty$ of $t$ such that $-t'\tau$ has SSP.1 and SSP.2 for all $0 \leqslant t' \leqslant t$, satisfies $t_\infty > h_{\mathrm{top}}(\phi_1)$. By Remark 3.3.11 and Corollary 3.2.6, for every small enough $|t|$, $-t\tau$ has SSP.1 and SSP.2 (thus $t_\infty > 0$), and the case $t = h_{\mathrm{top}}(\phi_1)$ corresponds to measures of maximal entropy for the billiard flow.

## 3.2 Topological Pressure, Variational Principle and Abramov Formula

In this section, we formulate definitions of topological pressure for the billiard map, and prove that – under some conditions – they are equivalent. Using a classical estimate, we then prove one direction of the variational principle. Finally, making use of the Abramov formula, we relate equilibrium states of $T$ with the ones of the billiard flow. More specifically, we identify the potential for $T$ which is related to – up to existence – the measures of maximal entropy of $\phi_t$.

We first introduce notation: Adopting the standard coordinates $x = (r, \varphi)$ on each connected component $M_i$ of

$$M := \partial Q \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = \bigsqcup_{i=1}^{D} \partial B_i \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where $r$ denotes arclength along $\partial B_i$, $\varphi$ is the angle the post-collisional trajectory makes with the normal to $\partial B_i$ and $M_i = \partial B_i \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In these coordinates, the collision map $T : M \to M$ preserve a smooth invariant probability measure $\mu_{\mathrm{SRB}}$ given by $\mathrm{d}\mu_{\mathrm{SRB}} = (2|\partial Q|)^{-1} \cos\varphi \, \mathrm{d}r \mathrm{d}\varphi$.

We now define the sets where $T$ and its iterates are discontinuous. Let $\mathcal{S}_0 \coloneqq \{(r,\varphi) \in M \mid |\varphi| = \pi/2\}$ denote the set of grazing collisions. For each nonzero $n \in \mathbb{N}$, let

$$\mathcal{S}_{\pm n} \coloneqq \bigcup_{i=0}^{n} T^{\mp i}\mathcal{S}_0,$$

denote the singularity set for $T^{\pm n}$. It would be natural to study the map $T$ restricted to the invariant set $M \setminus \cup_{n\in\mathbb{Z}}\mathcal{S}_n$ where $T$ is continuous, however the set of curves $\cup_{n\in\mathbb{Z}}\mathcal{S}_n$ is dense in $M$ [CM06, Lemma 4.55]. We thus introduce the classical family of partitions of $M$ as follows.

For $k$, $n \geqslant 0$, let $\mathcal{M}_{-k}^{n}$ denote the partition of $M \setminus (\mathcal{S}_{-k} \cup \mathcal{S}_n)$ into its maximal connected components. Note that all elements of $\mathcal{M}_{-k}^{n}$ are open sets on which $T^i$ is continuous, for all $-k \leqslant i \leqslant n$. Since the thermodynamic sums over elements of $\mathcal{M}_0^n$ of a potential $g$ will play a key role in the estimates on the norms of the iterates of the transfer operator $\mathcal{L}_g$ in Section 3.5, it should be natural – by analogy to the case of continuous maps – to define the topological pressure from these sums.

Another natural family of partitions is given as follows. Let $\mathcal{P}$ denote the partition of $M$ into maximal connected components on which both $T$ and $T^{-1}$ are continuous. Define $\mathcal{P}_{-k}^{n} = \bigvee_{i=-k}^{n} T^{-i}\mathcal{P}$ and remark that $T^i$ is continuous on every element of $\mathcal{P}_{-k}^{n}$, for all $-k \leqslant i \leqslant n$.

The interior of each element of $\mathcal{P}$ corresponds to precisely one element of $\mathcal{M}_{-1}^{1}$, but its refinements $\mathcal{P}_{-k}^{n}$ may also contain some isolated points – this happens if three or more scatterers have a common grazing collision. These partitions already appeared in the work of Baladi and Demers, where they proved [BD20, Lemma 3.2] that the number of isolated points in $\mathcal{P}_{-k}^{n}$ grows linearly in $n + k$.

Finally, denote $\mathring{\mathcal{P}}_{-k}^{n}$ the collection of interior of elements of $\mathcal{P}_{-k}^{n}$. In [BD20, Lemma 3.3], Baladi and Demers proved that $\mathring{\mathcal{P}}_{-k}^{n} = \mathcal{M}_{-k-1}^{n+1}$, for all $n$, $k \geqslant 0$. It should be natural that the topological pressures obtained from these three families of partitions coincide. This is the content of Theorem 3.2.1.

In order to formulate the result on the equivalence between definitions of topological pressure for $T$, we need to be more specific about the definition of *piecewise* Hölder.

We say that a function $g$ is $(\mathcal{M}_0^1, \alpha)$-Hölder, $0 < \alpha < 1$, if $g$ is $\alpha$-Hölder continuous on each element of the partition $\mathcal{M}_0^1$. We define the $C^\alpha$ norm $|g|_{C^\alpha}$ of $g$ to be the maximum, over all connected components $U$ of the domain of continuity of $g$, of the usual $C^\alpha$ norm $|g|_{C^\alpha(U)}$, that is

$$|g|_{C^\alpha} = \max\{|g|_{C^0(U)} + H_U^\alpha(g) \mid U \text{ connected set on which } g \text{ is continuous}\},$$

where

$$H_U^\alpha(g) = \sup_{x,y\in U} \frac{|g(x) - g(y)|}{d(x,y)^\alpha}.$$

Similarly, we say that a function $g$ is $\mathcal{M}_0^1$-continuous if $g$ is bounded and continuous on each element of the partition $\mathcal{M}_0^1$. We define the $C^0$ norm $|g|_{C^0}$ to be the maximum over all connected components $U$ of the domain of continuity of $g$, of the usual $C^0$ norm, that is

$$|g|_{C^0} = \max\{|g|_{C^0(U)} \mid U \text{ connected set on which } g \text{ is continuous}\}.$$

**Theorem 3.2.1.** *Let $g : M \to \mathbb{R}$ be a potential bounded from above. Then*

$$\lim_{n\to+\infty} \frac{1}{n} \log \sum_{A\in\mathcal{P}_0^n} \sup_{x\in A} e^{(S_n g)(x)} =: P_*(T,g)$$

*exists and is called the pressure of $g$. Moreover, the map $g \mapsto P_*(T,g)$ is convex.*

*When $g$ is $\mathcal{M}_0^1$-continuous and $P_*(T,g) - \sup g > 0$, the following limits exist and are equal to $P_*(T,g)$.*

*(i)* $\lim_{n\to+\infty} \frac{1}{n} \log \sum_{A\in\mathring{\mathcal{P}}_0^n} \sup_{x\in A} e^{(S_n g)(x)}$,

*(ii)* $\lim_{n\to+\infty} \frac{1}{n} \log \sum_{A\in\mathcal{M}_0^n} \sup_{x\in A} e^{(S_n g)(x)}$.

*Furthermore, when $g$ is also $(\mathcal{M}_0^1,\alpha)$-Hölder continuous, then the following limits are equal to $P_*(T,g)$.*

*(iii)* $\lim_{n\to+\infty} \frac{1}{n} \log \sum_{A\in\mathcal{P}_0^n} \inf_{x\in A} e^{(S_n g)(x)}$,

*(iv)* $\lim_{n\to+\infty} \frac{1}{n} \log \sum_{A\in\mathring{\mathcal{P}}_0^n} \inf_{x\in A} e^{(S_n g)(x)}$,

*(v)* $\lim_{n\to+\infty} \frac{1}{n} \log \sum_{A\in\mathcal{M}_0^n} \inf_{x\in A} e^{(S_n g)(x)}$,

*Finally, for a bounded potential, the sequence $n \mapsto \log \sum_{A\in\mathcal{M}_0^n} \sup_{x\in A} e^{(S_{n-1} g)(x)}$ is subadditive.*

**Proposition 3.2.2.** *Let $g$ be a $\mathcal{M}_0^1$-continuous potential. Let $P_{\mathrm{span}}(T,g)$ and $P_{\mathrm{sep}}(T,g)$ be the pressure obtained using Bowen's definition with, respectively, spanning sets and separating sets. Then $P_{\mathrm{span}}(T,g) \leqslant P_*(T,g)$ and $P_{\mathrm{sep}}(T,g) \leqslant P_*(T,g)$. When $P_*(T,g) - \sup g > 0$, then $P_*(T,g) = P_{\mathrm{sep}}(T,g)$. Furthermore, when $P_*(T,g) - \sup g > 0$ and $g$ is $(\mathcal{M}_0^1,\alpha)$-Hölder, $P_*(T,g) = P_{\mathrm{span}}(T,g)$.*

The proof of the last three forms of $P_*(T,g)$ in Theorem 3.2.1 relies crucially on the following lemma.

**Lemma 3.2.3.** *For every $(\mathcal{M}_0^1,\alpha)$-Hölder continuous potential $g$ there exists a constant $C_g$ such that for all $n \geqslant 1$ and all $P \in \mathcal{P}_0^n$,*

$$\sup_P e^{S_n g} \leqslant C_g \inf_P e^{S_n g}.$$

*The estimate still holds, for the same constant $C_g$, when $\mathcal{P}_0^n$ is replaced by $\mathcal{P}_{-l}^n$, $\mathring{\mathcal{P}}_{-l}^n$ or $\mathcal{M}_{-l}^n$, for any fixed $l \geqslant 0$.*

Before the proofs of these results, we first recall that $T$ is uniformly hyperbolic in the sense [CM06] that the cones

$$\begin{aligned} \mathcal{C}^u &:= \{(dr, d\varphi) \in \mathbb{R}^2 \mid \kappa_{\min} \leqslant d\varphi/dr \leqslant \kappa_{\max} + 1/\tau_{\min}\}, \\ \mathcal{C}^s &:= \{(dr, d\varphi) \in \mathbb{R}^2 \mid -\kappa_{\min} \geqslant d\varphi/dr \geqslant -\kappa_{\max} - 1/\tau_{\min}\}, \end{aligned} \tag{3.2.1}$$

are strictly invariant under $DT$ and $DT^{-1}$, respectively, whenever these derivatives exist. Here $\kappa_{\max}$ is the maximum curvature of the scatterer boundaries, $\kappa_{\min}$ the minimum, and

$\tau_{\min}$ is the minimum of the return time function $\tau$. Furthermore, there exists $C_1 > 0$ such that for all $n \geqslant 1$,

$$||D_x T^n(v)|| \geqslant C_1 \Lambda^n ||v||, \forall v \in \mathcal{C}^u, \quad ||D_x T^{-n}(v)|| \geqslant C_1 \Lambda^n ||v||, \forall v \in \mathcal{C}^s,$$

where $\Lambda = 1 + 2\kappa_{\min}\tau_{\min}$ is the minimum hyperbolicity constant.

*Proof of Lemma 3.2.3.* Let $d_n$ denote the Bowen distance, that is the dynamical distance given by

$$d_n(x,y) = \max_{0 \leqslant i \leqslant n} d(T^i x, T^i y),$$

where $d(x,y)$ is the Euclidean metric on each $M_i$, with $d(x,y) = 10D \max_i \operatorname{diam}(M_i)$ if $x$ and $y$ belong to different $M_i$ (this definition ensure we have a compact set). Let $\varepsilon_0 > 0$ be as in [BD20, eq (3.3)], that is: if $d_n(x,y) < \varepsilon_0$ then $x$ and $y$ lie in the same element of $\mathcal{M}_0^n$. Therefore, by the uniform hyperbolicity of $T$, if $d(T^i(x), T^i(y)) \leqslant \varepsilon_0/2$ for all $|i| \leqslant n$ then $d(x,y) \leqslant C_1 \Lambda^{-n} \varepsilon_0/2$.

Given a potential $g$, for all integers $m$, define the $m$-th variation by

$$\mathrm{Var}_m(g, T, \varepsilon) \coloneqq \sup\{|g(x) - g(y)| \mid d(T^j x, T^j y) \leqslant \varepsilon, |j| \leqslant m\}.$$

When $g$ is $(\mathcal{M}_0^1, \alpha)$-Hölder, we get that $\mathrm{Var}_m(g, T, \frac{\varepsilon_0}{2C_1}) \leqslant C(\frac{\varepsilon_0}{2}\Lambda^{-m})^\alpha$. Therefore

$$\sum_{m \geqslant 0} \mathrm{Var}_m\Big(g, T, \frac{\varepsilon_0}{2C_1}\Big) =: K < \infty.$$

By uniform hyperbolicity of $T$, there exists $k_\varepsilon$ such that $\operatorname{diam}(\mathcal{M}_{-k_\varepsilon}^{n+1}) < \varepsilon_0/2C_1$ for all $n \geqslant k_\varepsilon$. It then follows from the proof of [BD20, Lemma 3.5] that if $x$ and $y$ lie in the same element of $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$, then $d_n(x,y) \leqslant \varepsilon_0/2C_1$, for all $n \geqslant 0$.

Let $P \in \mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$ and let $x, y \in P$. Let $0 \leqslant k \leqslant n$. Then for all $|j| < m_k \coloneqq \min(k, n-k)$, $d(T^j(T^k x), T^j(T^k y)) < \varepsilon_0/2C_1$ and so $|g(T^k x) - g(T^k y)| \leqslant \mathrm{Var}_{m_k}(g, T, \frac{\varepsilon_0}{2C_1})$. Therefore

$$|S_n g(x) - S_n g(y)| \leqslant 2 \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor + 1} \mathrm{Var}_m\Big(g, T, \frac{\varepsilon_0}{2C_1}\Big) \leqslant 2K < \infty.$$

Now, let $P \in \mathcal{P}_0^n$ for some $n \geqslant 2k_\varepsilon$. Notice that $\mathcal{P}_0^n = \bigvee_{i=k_\varepsilon}^{n-k_\varepsilon} T^{-i}\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}$, in other words for all $l$ such that $k_\varepsilon \leqslant l \leqslant n - k_\varepsilon$, $T^l P$ is included in an element of $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon}$. Finally, by decomposing each orbit into three parts, we get that for all $x, y \in P$,

$$\begin{aligned} e^{S_n g(x) - S_n g(y)} &= e^{S_{k_\varepsilon} g(x) - S_{k_\varepsilon} g(y)} e^{S_{n-2k_\varepsilon} g(T^{k_\varepsilon} x) - S_{n-2k_\varepsilon} g(T^{k_\varepsilon} y)} e^{S_{k_\varepsilon} g(T^{n-k_\varepsilon} x) - S_{k_\varepsilon} g(T^{n-k_\varepsilon} y)} \\ &\leqslant e^{2k_\varepsilon(\sup g - \inf g)} e^{2K}. \end{aligned}$$

The claim holds for $n \geqslant 2k_\varepsilon$ by taking the sup over $x$ and the inf over $y$ in $P$. Since there are only finitely many values of $n$ to correct for, by taking a larger constant, the claimed estimate holds for all $n \geqslant 1$.

Fix some $l \geqslant 0$. Since an element $P \in \mathcal{P}_{-l}^n$ is contained in a unique element $\tilde{P} \in \mathcal{P}_0^n$, we get that

$$\sup_P e^{S_n g} \leqslant \sup_{\tilde{P}} e^{S_n g} \leqslant C \inf_{\tilde{P}} e^{S_n g} \leqslant C \inf_P e^{S_n g}.$$

Now, assume that $\mathring{P} \neq \emptyset$. Then, by the continuity of $S_n g$ on $P$, the estimate also holds when the sup and the inf are taken over $\mathring{P}$. In other words, the claim is true for all $P \in \mathring{\mathcal{P}}_l^n$.

Since by [BD20, Lemma 3.3], $\mathring{\mathcal{P}}_{-l}^n = \mathcal{M}_{-l-1}^{n+1}$, the claim is true for all $P \in \mathcal{M}_{-l}^n$, for fixed $l \geqslant 1$. We finish the proof with the case $P \in \mathcal{M}_0^n$. Remark that letting $A \in \mathcal{M}_{-1}^n$, then $T^{-1}A \in \mathcal{M}_0^{n+1}$. Therefore

$$\begin{aligned} e^{-\sup g} \sup_{T^{-1}A} e^{S_{n+1}g} &\leqslant \sup_{T^{-1}A} e^{S_{n+1}g-g} = \sup_A e^{S_n g} \leqslant C \inf_A e^{S_n g} = C \inf_{T^{-1}A} e^{S_{n+1}g-g} \\ &\leqslant 2Ce^{-\inf g} \inf_{T^{-1}A} e^{S_{n+1}g}. \end{aligned}$$

Only in this last case, we need to replace $C$ by $2Ce^{\sup g - \inf g} \geqslant C$. □

*Proof of Theorem 3.2.1.* Let $p_n = \sum_{A \in \mathcal{P}_0^n} \sup_{x \in A} e^{(S_n g)(x)}$. Then, for $k \geqslant n$,

$$p_{n+k} = \sum_{B \cap C \in \mathcal{P}_0^n \bigvee T^{-n}\mathcal{P}_0^k} \sup_{x \in B \cap C} e^{(S_n g)(x) + (S_k g)(T^n x)} \leqslant p_n \, p_k.$$

Therefore $(\log p_n)_n$ is a sub-additive sequence. It is then classical that $\frac{1}{n} \log p_n$ converges to $\inf_{n \geqslant 1} \frac{1}{n} \log p_n$, hence $P_*(T, g)$ exists. We now prove the statement about convexity. Let $g_1$ and $g_2$ be two potentials bounded from above and $p \in [0, 1]$. Using the Hölder inequality, we get that for all $n \geqslant 1$

$$\begin{aligned} \sum_{A \in \mathcal{P}_0^n} \sup_A e^{p\, S_n g_1 + (1-p) S_n g_2} &\leqslant \sum_{A \in \mathcal{P}_0^n} \sup_A \left(e^{S_n g_1}\right)^p \left(\sup_A e^{S_n g_2}\right)^{1-p} \\ &\leqslant \Big(\sum_{A \in \mathcal{P}_0^n} \sup_A e^{S_n g_1}\Big)^p \Big(\sum_{A \in \mathcal{P}_0^n} \sup_A e^{S_n g_2}\Big)^{1-p}. \end{aligned}$$

Taking the appropriate limits, we get that $P_*(T, pg_1 + (1-p)g_2) \leqslant pP_*(T, g_1) + (1-p)P_*(T, g_2)$, hence the claimed convexity.

For $(i)$, consider $\tilde{p}_n = \sum_{A \in \mathring{\mathcal{P}}_0^n} \sup_{x \in A} e^{(S_n g)(x)}$. Notice that

$$\mathcal{P}_0^n = \{A \in \mathcal{P}_0^n \mid \mathring{A} \neq \emptyset\} \sqcup \{A \in \mathcal{P}_0^n \mid \mathring{A} = \emptyset\}.$$

Now, Baladi and Demers proved in [BD20, Lemma 3.2] that the cardinality of the second term in the right hand side grows at most linearly. Hence

$$\sum_{\substack{A \in \mathcal{P}_0^n \\ \mathring{A} = \emptyset}} \sup_{x \in A} e^{(S_n g)(x)} \leqslant Cne^{n \sup g}.$$

By the smoothness of $g$,

$$\sum_{\substack{A \in \mathcal{P}_0^n \\ \mathring{A} \neq \emptyset}} \sup_{x \in A} e^{(S_n g)(x)} = \sum_{A \in \mathring{\mathcal{P}}_0^n} \sup_{x \in A} e^{(S_n g)(x)}.$$

Thanks to the assumption $P_*(T, g) - \sup g > 0$, the sum over elements $A \in \mathcal{P}_0^n$ with $\mathring{A} \neq \emptyset$ dominates the sum over $A$ with $\mathring{A} = \emptyset$. Thus $(\frac{1}{n} \log \tilde{p}_n)_n$ converges to the same limit as $(\frac{1}{n} \log p_n)_n$ does.

For $(ii)$, we use [BD20, Lemma 3.3] that $\mathring{\mathcal{P}}_0^n = \mathcal{M}_{-1}^{n+1}$. Hence

$$\sum_{A\in\mathcal{M}_0^{n+1}} \sup_{x\in A} e^{(S_{n+1}g)(x)} \leqslant \sum_{A\in\mathcal{M}_{-1}^{n+1}} \sup_{x\in A} e^{(S_{n+1}g)(x)} \leqslant \sum_{A\in\mathring{\mathcal{P}}_0^n} \sup_{x\in A} e^{(S_n g)(x)} \sup_{x\in M} e^{g(x)}.$$

Furthermore, since $\mathcal{M}_{-1}^{n+1} = \mathcal{M}_0^{n+1} \bigvee \mathcal{M}_{-1}^0$, each element of $\mathcal{M}_0^{n+1}$ contains at most $\#\mathcal{M}_{-1}^0$ elements of $\mathcal{M}_{-1}^{n+1}$. Hence

$$\sum_{A\in\mathcal{M}_0^{n+1}} \sup_{x\in A} e^{(S_{n+1}g)(x)} \geqslant \frac{1}{\#\mathcal{M}_{-1}^0} \sum_{A\in\mathring{\mathcal{P}}_0^n} \sup_{x\in A} e^{(S_n g)(x)} \inf_{x\in M} e^{g(x)}.$$

Point $(iii)$ (resp. $(iv)$, $(v)$) follows directly from the definition of $P_*(T,g)$ (resp. from point $(i)$, $(ii)$) and from Lemma 3.2.3 since

$$\inf_A e^{S_n g} \leqslant \sup_A e^{S_n g} \leqslant C \inf_A e^{S_n g},$$

for all $A$ in $\mathcal{P}_0^n$ (resp. $\mathring{\mathcal{P}}_0^n$, $\mathcal{M}_0^n$). For the final claim, we prove that $\log \sum_{P\in\mathring{\mathcal{P}}_1^n} \sup_P e^{S_n g}$ is subadditive. Take $P$ a nonempty element of $\mathring{\mathcal{P}}_1^{n+m}$. It is the interior of an intersection of elements of the form $T^{-j}A_j$ for some $A_j \in \mathcal{P}$, for $j = 1, \ldots, n+m$. This is equal to the intersection of the interiors of $T^{-j}A_j$. But since $P$ is nonempty, none of the $T^{-j}A_j$ has empty interior, and so none of the $A_j$ has empty interior. Thus the interiors of $A_j$ are in $\mathring{\mathcal{P}}$. Now, splitting the intersection of the first $n$ sets from the last $m$, we see that the intersection of the first $n$ sets forms an element of $\mathring{\mathcal{P}}_1^n$. For the last $m$ sets, we can factor out $T^{-n}$ at the price of making the set slightly bigger:

$$\mathrm{int}(T^{-n-j}A_{-n-j}) \subset T^{-n}(\mathrm{int}(T^{-j}(A_{-n-j}))), \quad 1 \leqslant j \leqslant m$$

where int denotes the interior of a set. Thus

$$\begin{aligned}
\sum_{P\in\mathring{\mathcal{P}}_1^{n+m}} \sup_P e^{S_{n+m}g} &\leqslant \sum_{\substack{A_{-j}\in\mathring{\mathcal{P}} \\ 1\leqslant j\leqslant n+m}} \sup\{e^{S_n g + S_m g\circ T^n}(x) \mid x \in \bigcap_{j=1}^n T^{-j}A_{-j} \cap T^{-n} \bigcap_{j=1}^m T^{-j}A_{-n-j}\} \\
&\leqslant \sum_{\substack{A_{-j}\in\mathring{\mathcal{P}} \\ 1\leqslant j\leqslant n}} \sup\{e^{S_n g}(x) \mid x \in \bigcap_{j=1}^n T^{-j}A_{-j}\} \sum_{\substack{A_{-j}\in\mathring{\mathcal{P}} \\ 1\leqslant j\leqslant m}} \sup\{e^{S_m g}(x) \mid x \in \bigcap_{j=1}^m T^{-j}A_{-j}\} \\
&\leqslant \sum_{P\in\mathring{\mathcal{P}}_1^n} \sup_P e^{S_n g} \sum_{P\in\mathring{\mathcal{P}}_1^m} \sup_P e^{S_m g}
\end{aligned}$$

Taking logs, the sequence is subadditive. And then so is the sequence with $\mathcal{M}_0^n$ in place of $\mathring{\mathcal{P}}_1^{n-1}$. □

*Proof of Proposition 3.2.2.* We first prove the claim about separating sets. Let $\varepsilon > 0$ and let $k_\varepsilon$ be large enough so that $\mathrm{diam}^s(\mathcal{M}_{-k_\varepsilon-1}^0) \leqslant C\Lambda^{-k_\varepsilon} < c_1\varepsilon$ for some constant $c_1$ to be defined later. Therefore $\mathrm{diam}^u(\mathcal{M}_{-k_\varepsilon-1}^{n+1}) \leqslant C\Lambda^{-k_\varepsilon} < c_1\varepsilon$ for all $n \geqslant k_\varepsilon$. By the uniform transversality between the stable and the unstable cones, we can choose $c_1$ such that $\mathrm{diam}(\mathcal{M}_{-k_\varepsilon-1}^{n+1}) < \varepsilon$ for all $n \geqslant k_\varepsilon$.

Let $E$ be $(n,\varepsilon)$-separated, for some $n \geqslant k_\varepsilon$. It is shown in the proof of [BD20, Lemma 3.4] that if $x, y \in E$ are distinct, then they cannot be contained in the same element of $\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}$.

Thus

$$\sum_{x\in E} e^{S_n g(x)} \leqslant \sum_{A\in\mathcal{P}_{-k_\varepsilon}^{k_\varepsilon+n}} |e^{S_n g}|_{C^0(A)} = \sum_{A\in\mathcal{P}_0^{2k_\varepsilon+n}} |e^{S_n g\circ T^{k_\varepsilon}}|_{C^0(A)}$$
$$\leqslant e^{k_\varepsilon(\sup g-\inf g)} \sum_{A\in\mathcal{P}_0^{2k_\varepsilon+n}} |e^{S_{2k_\varepsilon+n} g}|_{C^0(A)}.$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}\log\sup\{\sum_{x\in E} e^{S_n g(x)} \mid \ E \text{ is } (n,\varepsilon)\text{-separated}\} \leqslant P_*(T,g) \quad , \text{for all } \varepsilon>0.$$

Taking the limit $\varepsilon\to 0$, we get $P_{\text{sep}}(T,g)\leqslant P_*(T,g)$.

For the reverse inequality, assume that $g$ is such that $P_*(T,g)-\sup g>0$. From the proof of [BD20, Lemma 3.4], there exists $\varepsilon_0>0$ such that for all $\varepsilon<\varepsilon_0$, any set $E$ which contains only one point per element of $\mathcal{M}_0^n$ is $(n,\varepsilon)$-separated. For all $A\in\mathcal{M}_0^n$, there exists $x\in A$ such that $e^{S_n g(x)}\geqslant \frac{9}{10}\sup_A e^{S_n g}$. Let $E$ be the collection of such $x$. Thus

$$\sum_{x\in E} e^{S_n g(x)} \geqslant \frac{9}{10}\sum_{A\in\mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)}\,.$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\log\sup\{\sum_{x\in E} e^{S_n g(x)} \mid \ E \text{ is } (n,\varepsilon)\text{-separated}\} \geqslant P_*(T,g)\,, \quad \text{for all } 0<\varepsilon<\varepsilon_0.$$

Taking the limit $\varepsilon\to 0$, we get $P_{\text{sep}}(T,g)\geqslant P_*(T,g)$, thus the claimed equality.

We now prove the claim concerning spanning sets. Let $\varepsilon>0$ and let $k_\varepsilon$ be such that $\operatorname{diam}(\mathcal{M}_{-k_\varepsilon-1}^{n+1})<\varepsilon$ for all $n\geqslant k_\varepsilon$. Let $F$ be a set containing one point in each element of $\mathcal{P}_{-k_\varepsilon}^{n+1}$. From the proof of [BD20, Lemma 3.5], $F$ is $(n,\varepsilon)$-spanning. Since

$$\sum_{x\in F} e^{S_n g(x)} \leqslant e^{k_\varepsilon(\sup g-\inf g)} \sum_{A\in\mathcal{P}_0^{2k_\varepsilon+n}} |e^{S_{2k_\varepsilon+n} g}|_{C^0(A)}$$

we get that

$$\lim_{n\to\infty}\frac{1}{n}\log\inf\{\sum_{x\in F} e^{S_n g(x)} \mid F \text{ is } (n,\varepsilon)\text{-spanning}\} \leqslant P_*(T,g)\,, \quad \text{for all } \varepsilon>0\,.$$

Taking the limit $\varepsilon\to 0$, we get $P_{\text{span}}(T,g)\leqslant P_*(T,g)$.

For the reverse inequality, assume that $g$ is a $(\mathcal{M}_0^1,\alpha)$-Hölder potential such that $P_*(T,g)-\sup g>0$. Let $\varepsilon<\varepsilon_0$ and let $F$ be a $(n,\varepsilon)$-spanning set. By the proof of [BD20, Lemma 3.5], each element of $\mathcal{M}_0^n$ contains at least one element of $F$. Thus

$$\sum_{x\in F} e^{S_n g(x)} \geqslant \sum_{A\in\mathcal{M}_0^n} \inf_A e^{S_n g}\,.$$

Taking the appropriate limits, we get that $P_{\text{span}}(T,g)\geqslant P_*(T,g)$, thus the claimed equality.

□

### 3.2.1 Easy Direction of the Variational Principle for the Pressure

Recall that given a $T$-invariant probability measure $\mu$ and a finite measurable partition $\mathcal{A}$ of $M$, the entropy of $\mathcal{A}$ with respect to $\mu$ is defined by $H_\mu(\mathcal{A}) = -\sum_{A\in\mathcal{A}} \mu(A)\log\mu(A)$, and the entropy of $T$ with respect to $\mathcal{A}$ is $h_\mu(T,\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{A}\Big)$.

**Lemma 3.2.4.** *Let $\varphi : M \to \mathbb{R}$ be a measurable function. Then*

$$P_*(T,\varphi) \geqslant P(T,\varphi) := \sup\{h_\mu(T) + \int \varphi \,\mathrm{d}\mu \mid \mu \text{ is a } T\text{-invariant Borel probability measure}\}$$

*Proof.* Let $\mu$ be a $T$-invariant probability measure on $M$. Notice that $\mathcal{P}$ is a generator for $T$ since $\bigvee_{i=-\infty}^{\infty} T^{-i}\mathcal{P}$ separates points in $M$. Thus $h_\mu(T) = h_\mu(T,\mathcal{P})$ (see for example [Wal82, Theorem 4.17]). Then,

$$\begin{aligned} h_\mu(T) + \int \varphi \,\mathrm{d}\mu &= \lim_{n\to\infty} \frac{1}{n} \sum_{A\in\mathcal{P}_0^n} \Big(-\mu(A)\log\mu(A) + \int_A S_n\varphi \,\mathrm{d}\mu\Big) \\ &\leqslant \lim_{n\to\infty} \frac{1}{n} \sum_{A\in\mathcal{P}_0^n} \mu(A)(\sup_A(S_n\varphi) - \log\mu(A)) \\ &\leqslant \lim_{n\to\infty} \frac{1}{n} \log \sum_{A\in\mathcal{P}_0^n} \sup_A e^{S_n\varphi} \leqslant P_*(T,\varphi) \end{aligned}$$

where we used [Wal82, Lemma 9.9] for the second inequality. □

### 3.2.2 Abramov Formula and Choice of the Potential $g$

In order to obtain the existence of MME for the billiard flow, we make use of the Abramov formula to relate equilibrium measure for $T$ and some potential $g$, to the MME of the flow. First, we need the following (classical) lemma.

**Lemma 3.2.5.** *Let $\varphi$ be a bounded non-negative measurable function such that $\varphi_0 := \inf\{\int \varphi \,\mathrm{d}\mu \mid T_*\mu = \mu\} > 0$. Then, there exists a unique real number $c_\varphi$ such that $P(T, -c_\varphi\varphi) = 0$.*

*Proof.* We first prove that the function $t \mapsto P(T, t\varphi)$ is increasing. Let $\varepsilon > 0$ and $t_1 < t_2$. There exists a $T$-invariant probability measure $\mu_1$ such that

$$P(T, t_1\varphi) \leqslant h_{\mu_1}(T) + t_1 \int \varphi \,\mathrm{d}\mu_1 + \varepsilon \leqslant P(T, t_2\varphi) - (t_2 - t_1)\varphi_0 + \varepsilon.$$

By this computation, we also get that $\lim_{t\to\pm\infty} P(T, t\varphi) = \pm\infty$ .

Now we prove that $t \mapsto P(T, t\varphi)$ is continuous. Let $\varepsilon > 0$ and $t \in \mathbb{R}$. By the previous computation, we get that $\varepsilon\varphi_0 \leqslant P(T, (t+\varepsilon)\varphi) - P(T, t\varphi)$. Let $\mu_2$ be such that $P(T, (t+\varepsilon)\varphi) \leqslant h_{\mu_2}(T) + (t+\varepsilon)\int \varphi \,\mathrm{d}\mu_2 + \varepsilon$. Thus

$$P(T, (t+\varepsilon)\varphi) - P(T, t\varphi) \leqslant \varepsilon(1 + \sup\varphi).$$

Therefore $t \mapsto P(T, t\varphi)$ is (strictly) increasing and continuous, so it must vanish at exactly one point, noted $-c_\varphi$. □

We can now use this lemma with the Abramov formula to get the following

**Corollary 3.2.6.** *Equilibrium measures of $T$ under the potential $-h_{top}(\phi_1)\tau$ and MME of the billiard flow (seen as a suspension flow) are in one-to-one correspondence through the bijection $\mu \mapsto \mu_\tau := \frac{1}{\mu(\tau)}\mu \otimes \lambda$, where $\lambda$ is the Lebesgue measure.*

*Proof.* Since $\tau \geqslant \tau_{\min} > 0$, the assumption of Lemma 3.2.5 is satisfied for $\varphi = \tau$. Let $c$ be the constant given by Lemma 3.2.5 such that $0 = P(T, -c\tau)$. Then, for every equilibrium state $m$ of $T$ under the potential $-c\tau$, we get

$$0 = h_m(T) - c\int \tau \, \mathrm{d}m \geqslant h_\mu(T) - c\int \tau \, \mathrm{d}\mu,$$

for all $T$-invariant measure $\mu$. Thus

$$c = \frac{h_m(T)}{\int \tau \, \mathrm{d}m} \geqslant \frac{h_\mu(T)}{\int \tau \, \mathrm{d}\mu}.$$

Now, by the Abramov formula, $c = h_{m_\tau}(\phi_1) \geqslant h_{\mu_\tau}(\phi_1)$. In other words, $m_\tau$ is a MME for the billiard flow. Furthermore, since $\phi_1$ is a continuous map of a compact metric space, by [Wal82, Theorem 8.6], we get that $h_{\mathrm{top}}(\phi_1) = \sup\{h_\mu(\phi_1) \mid (\phi_1)_*\mu = \mu\}$. Thus $c = h_{\mathrm{top}}(\phi_1)$.

To prove that the map is onto, we use that any $\phi_t$-invariant probability measure $\mu_\tau$ must be of the form $\frac{1}{\mu(\tau)}\mu \otimes \lambda$, for some $T$-invariant probability measure $\mu$. Thus, reversing the above computations, we get that if $\mu_\tau$ is a MME, then $\mu$ is an equilibrium state for $T$ under the potential $-h_{\mathrm{top}}(\phi_1)\tau$. □

Therefore, proving the existence and uniqueness of the MME for the billiard flow is equivalent to proving the existence and uniqueness of the equilibrium state of $T$ under the potential $g = -h_{top}(\phi_1)\tau$. Notice that in the second case, $g$ is $(\mathcal{M}_0^1, \frac{1}{2})$-Hölder continuous and the condition $P_*(T, g) - \sup g > 0$ from Theorem 3.2.1 is realised since $P_*(T, g) - \sup g \geqslant P(T, -h_{top}(\phi_1)\tau) + h_{top}(\phi_1)\tau_{\min} > 0$.

*Remark* 3.2.7. Using similar arguments as in Corollary 3.2.6, we can relate the equilibrium states of $\phi_t$ under the (measurable) potential $\tilde{g} : \Omega \to \mathbb{R}$ to the ones of $T$ under $g = \lambda(\tilde{g}) - P(\phi_1, \tilde{g})\tau$, where $\lambda(\tilde{g}) : M \to \mathbb{R}$ is given by

$$\lambda(\tilde{g})(x) = \int_0^{\tau(x)} \tilde{g}(\phi_t(x)) \, \mathrm{d}t.$$

## 3.3 Growth Lemma and Fragmentation Lemmas

This section contains growth lemmas, controlling the growth in complexity of the iterates of a stable curve, with a weight $g$. We also formulate the precise definitions of the conditions SSP.1 and SSP.2. The first condition will be used to prove the “Lasota–Yorke" bounds on the transfer operator $\mathcal{L}_g$ in Proposition 3.5.1, as well as the lower bound on the spectral radius in Theorem 3.5.3, while SSP.2 will be crucial for the absolute continuity (Corollary 3.6.8) used to prove statistical properties (Propositions 3.6.12 and 3.6.15) and to compute the pressure (Corollary 3.6.14).

In view of deriving Lemma 3.3.4 from Lemma 3.3.3, we first need to introduce a class of curves more general than stable manifolds. Recall the stable and unstable cones (3.2.1).

First, denote by $\mathcal{W}^s$ the set of all nontrivial connected subsets $W$ of stable manifolds for $T$ so that $W$ has length at most $\delta_0$. Such curves have curvature bounded above by a fixed constant [CM06, Prop. 4.29]. Thus $T^{-1}\mathcal{W}^s = \mathcal{W}^s$, up to subdivision of curves. We define $\mathcal{W}^u$ similarly from unstable manifolds of $T$.

Now, we define a set of cone-stable curves $\widehat{\mathcal{W}}^s$ whose tangent vectors all lie in the stable cone $\mathcal{C}^s$ for the map, with length at most $\delta_0$ (to be determined latter) and curvature bounded above so that $T^{-1}\widehat{\mathcal{W}}^s \subset \widehat{\mathcal{W}}^s$, up to subdivision of curves. We define a set of cone-unstable curves $\widehat{\mathcal{W}}^u$ similarly. These sets of curves will be relevant since $\mathcal{S}_n$ and $\mathcal{S}_{-k}$ are composed of curves in $\widehat{\mathcal{W}}^s$ and $\widehat{\mathcal{W}}^u$, respectively. Obviously, $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$.

For $\delta \in (0, \delta_0]$ and $W \in \widehat{\mathcal{W}}^s$, let $\mathcal{G}_0^\delta(W) := \{W\}$. For $n \geqslant 1$, define the *$\delta$-scaled subdivision* $\mathcal{G}_n^\delta(W)$ inductively as the collection of smooth components of $T^{-1}(W')$ for $W' \in \mathcal{G}_{n-1}^\delta(W)$, where elements longer than $\delta$ are subdivided to have length between $\delta/2$ and $\delta$. Thus $\mathcal{G}_n^\delta(W) \subset \widehat{\mathcal{W}}^s$ for each $n$ and $\cup_{U \in \mathcal{G}_n^\delta(W)} U = T^{-n}W$. Moreover, if $W \in \mathcal{W}^s$, then $\mathcal{G}_n^\delta(W) \subset \mathcal{W}^s$.

Denote by $L_n^\delta(W)$ those elements of $\mathcal{G}_n^\delta(W)$ having length at least $\delta/3$ (the *long* curves), $S_n^\delta(W) := \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$ (the *short* curves), and define $\mathcal{I}_n^\delta(W)$ to comprise those elements $U \in \mathcal{G}_n^\delta(W)$ for which $T^iU$ is contained in an element of $S_{n-i}^\delta(W)$ for all $0 \leqslant i \leqslant n-1$.

A fundamental fact [Che01, Lemma 5.2] we will use is that the growth in complexity for the billiard is at most linear:

$$\exists\ K > 0 \ \text{ such that } \ \forall\ n \geqslant 0, \text{ the number of curves in } \mathcal{S}_{\pm n} \text{ that intersect at a single point is at most } Kn. \tag{3.3.1}$$

### 3.3.1 Growth Lemma

**Lemma 3.3.1.** *For any $m \in \mathbb{N}$, there exists $\delta_0 = \delta_0(m) \in (0,1)$ such that for all $W \in \widehat{\mathcal{W}}^s$, if $|W| < \delta_0$, then for all $0 \leqslant l \leqslant 2m$, $T^{-l}W$ comprises at most $Km+1$ connected components.*
*Furthermore, for any $\delta \in (0, \delta_0]$, the $\delta$-scaled subdivisions satisfy the following estimates: for all $n \geqslant 1$, all $\gamma \in [0, \infty)$, all $W \in \widehat{\mathcal{W}}^s$, and all $\mathcal{M}_0^1$-continuous potential $g$, we have*

$$a)\quad \sum_{W_i \in \mathcal{I}_n^\delta(W)} \left(\frac{\log|W|}{\log|W_i|}\right)^\gamma |e^{S_n g}|_{C^0(W_i)} \leqslant 2^{((n\vee n_0)s_0+1)\gamma+1}(Km+1)^{n/m} e^{n \sup g}$$

$$b)\quad \sum_{W_i \in \mathcal{G}_n^\delta(W)} \left(\frac{\log|W|}{\log|W_i|}\right)^\gamma |e^{S_n g}|_{C^0(W_i)} \leqslant \min\Bigg\{ 2C\delta^{-1} 2^{((n\vee n_0)s_0+1)\gamma} \sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)}\,,$$
$$2^{2\gamma+1} C\delta^{-1} \sum_{j=1}^{n} 2^{(j\vee n_0)s_0\gamma}(Km+1)^{j/m} e^{j \sup g} \sum_{A \in \mathcal{M}_0^{n-j}} |e^{S_{n-j} g}|_{C^0(A)} \Bigg\}$$

*where $(n \vee n_0) = \max(n, n_0)$.*
*Moreover, if $|W| \geqslant \delta/2$, then both factors $2^{(ns_0+1)\gamma}$ can be replaced by $2^\gamma$.*

*Proof.* By [CM06, Exercise 4.50], there exist constants $\delta_{\text{CM}} > 0$ and $C \geqslant 1$ such that for all $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta_{\text{CM}}$, then $|T^{-1}W| \leqslant C|W|^{1/2}$. Notice also that there exists $\Lambda_1 := \Lambda_1(\varphi_0)$ such that for $W \in \widehat{\mathcal{W}}^s$ with $T^{-1}W \cap \{|\varphi| > \varphi_0\} = \emptyset$, then $|T^{-1}W| \leqslant \Lambda_1|W|$. We want to iterate these bounds.

Let $\delta \in (0, \delta_{\text{CM}}]$, $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta$, and $W_i \in \mathcal{I}_n^\delta(W)$. Let $V \subset W$ corresponding to $W_i$, that is $V = T^nW_i$. Thus, for all $1 \leqslant j \leqslant n$, we have $|T^{-j}V| = |T^{n-j}W_i| \leqslant \delta/3$.

We can decompose $V = \bigsqcup_{i_0 \in I_0} V^0_{i_0,\mathrm{graz}} \cup \bigsqcup_{j_0 \in J_0} V^0_{j_0,\exp}$ such that: for all $i_0 \in I_0$, $T^{-1}V^0_{i_0,\mathrm{graz}} \subset \{|\varphi| \geqslant \varphi_0\}$, and thus $|T^{-1}V^0_{i_0,\mathrm{graz}}| \leqslant C|V^0_{i_0,\mathrm{graz}}|^{1/2}$, and for all $j_0 \in J_0$, $T^{-1}V^0_{j_0,\exp} \subset \{|\varphi| < \varphi_0\}$, and thus $|T^{-1}V^0_{j_0,\exp}| \leqslant \Lambda_1 |V^0_{j_0,\exp}|$. We can perform the same decomposition for $V^0_{i_0,\mathrm{graz}}$ or $V^0_{j_0,\exp}$ instead of $V$:

$$V^0_{i_0,\mathrm{graz}} = \bigsqcup_{i_1} V^{1,i_0}_{i_1,\mathrm{graz}} \cup \bigsqcup_{j_1} V^{1,i_0}_{j_1,\exp}\ , \qquad V^0_{j_0,\exp} = \bigsqcup_{i_1} V^{1,j_0}_{i_1,\mathrm{graz}} \cup \bigsqcup_{j_1} V^{1,j_0}_{j_1,\exp}.$$

We can iterate this decomposition until having a decomposition of $T^{-n}V = W_i$. Notice that since the stable curves $T^{-j}V$ have length at most $\delta/3 \leqslant \delta_{\mathrm{CM}}/3$ and are uniformly transverse to $\mathcal{S}_0$, they can cross $\{|\varphi| \geqslant \varphi_0\}$ at most $B$ times, where $B > 0$ is a constant uniform in $W$. Thus, at each step, a curve is split into at most $2B$ pieces.

Thus $W_i = T^{-n}V = \bigsqcup_{\substack{* \in \{\mathrm{graz},\exp\} \\ \alpha_k \in I_k \sqcup J_k}} V^{n,\alpha_0,\dots,\alpha_{n-1}}_{\alpha_n,*}$, where the union is made of at most $(2B)^n$ elements we can estimate the length.

Consider first the case $n \leqslant n_0$. By definition, $s_0$ is such that $s_0 = \sup_M \frac{1}{n_0} \sum_{k=0}^{n_0-1} \mathbb{1}_{\{|\varphi|>\varphi_0\}} \circ T^k < 1$. Thus, for each $V^{n,\alpha_0,\dots,\alpha_{n-1}}_{\alpha_n,*}$ there are at most $s_0 n_0$ indices $\alpha_k \in I_k$. Thus $|V^{n,\alpha_0,\dots,\alpha_{n-1}}_{\alpha_n,*}| \leqslant C^2 \Lambda_1^{n_0} |V|^{2^{-s_0 n_0}}$. Therefore

$$|W_i| = |T^{-n_0}V| \leqslant (2B)^{n_0} C^2 \Lambda_1^{n_0} |V|^{2^{-s_0 n_0}} \leqslant \tilde{C}|W|^{2^{-s_0 n_0}}\ , \quad \forall\, W_i \in \mathcal{I}^\delta_n(W),\ n \leqslant n_0,\ \delta \leqslant \delta_{\mathrm{CM}} \tag{3.3.2}$$

Now, consider the case $n = kn_0 + l$, for $k \geqslant 1$ and $0 \leqslant l < n_0$. By construction, if $W_i \in \mathcal{I}^\delta_n(W)$, then $T^l W_i \subset W^0_i \in \mathcal{I}^\delta_{kn_0}(W)$ and $T^{n_0} W^j_i \subset W^{j+1}_i \in \mathcal{I}^\delta_{(k-j-1)n_0}(W)$ for all $0 \leqslant j \leqslant k-1$. Thus, we can iterate (3.3.2):

$$|W_i| \leqslant \tilde{C}|W^0_i|^{2^{-s_0 n_0}} \leqslant \tilde{C}^{\sum_{m=0}^{j} 2^{-m s_0 n_0}} |W^j_i|^{2^{-j s_0 n_0}} \leqslant \tilde{C}^2 |W|^{2^{-(k+1)s_0 n_0}}$$

and so $|W_i| \leqslant \tilde{C}^2|W|^{2^{-ns_0}}$ for all $W_i \in \mathcal{I}^\delta_n(W)$, $n \geqslant n_0$ and all $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta_{\mathrm{CM}}$.

Therefore, if $\delta \leqslant \min(\tilde{C}^{-2}, \delta_{\mathrm{CM}})$, we have

$$\left(\frac{\log|W|}{\log|W_i|}\right)^\gamma \leqslant \left(2^{s_0 n}\left(1 - \frac{\log \tilde{C}^2}{\log|W_i|}\right)\right)^\gamma \leqslant 2^{(ns_0+1)\gamma},\ \forall\, W_i \in \mathcal{I}^\delta_n(W),$$

since $|W_i| \leqslant \delta$.

(a) Let $m \geqslant 1$ and $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta \leqslant \min(\tilde{C}^{-2}, \delta_{\mathrm{CM}})$. First, we want to estimate the number of smooth components of $T^{-l}W$, for $0 \leqslant l \leqslant 2m$. The problem is the same as knowing the number of connected components of $W \smallsetminus \mathcal{S}_{-l}$. Now, by (3.3.1), at most $Kl$ curves in $\mathcal{S}_{-l}$ can intersect at a given point. Since $W$ and $\mathcal{S}_{-l}$ are uniformly transverse, for each $0 \leqslant l \leqslant 2m$ there exists $\delta_{(l)}$ such that if $|W| < \delta_{(l)}$ then $W \smallsetminus \mathcal{S}_{-l}$ has at most $Kl+1$ connected components. Let $\delta_0 \coloneqq \min\{\delta_{(l)} \mid 0 \leqslant l \leqslant 2m\}$.

Let $n \geqslant 1$, $\delta \in (0, \delta_0]$ and $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta$. We want to estimate $\#\mathcal{I}^\delta_n(W)$. We prove by induction that $\#\mathcal{I}^\delta_{jm}(W) \leqslant (Km+1)^j$. For $j = 1$, this follows from the choice of $\delta_0$. Since elements of $\mathcal{I}^\delta_{(j+1)m}(W)$ are of the form $V \in \mathcal{I}^\delta_m(W_i)$ for $W_i \in \mathcal{I}^\delta_{jm}(W)$, we have

$$\#\mathcal{I}^\delta_{(j+1)m}(W) \leqslant (Km+1)\#\mathcal{I}^\delta_{jm}(W) \leqslant (Km+1)^{j+1}.$$

Now for estimating $\#\mathcal{I}^\delta_{jm+l}(W)$, $0 \leqslant l < m$, we only need to modify the last step:

$$\#\mathcal{I}^\delta_{jm+l}(W) \leqslant (K(m+l)+1)\#\mathcal{I}^\delta_{(j-1)m}(W) \leqslant 2(Km+1)^j.$$

Therefore, $\#\mathcal{I}_n^\delta(W) \leqslant 2(Km+1)^{n/m}$, for all $n \geqslant 1$.

Finally, we have that for $n \geqslant n_0$

$$\sum_{W_i \in \mathcal{I}_n^\delta(W)} \left(\frac{\log|W|}{\log|W_i|}\right)^\gamma |e^{S_n g}|_{C^0(W_i)} \leqslant e^{n \sup g} 2^{(ns_0+1)\gamma} \#\mathcal{I}_n^\delta(W)$$
$$\leqslant 2^{(ns_0+1)\gamma+1}(Km+1)^{n/m} e^{n \sup g}$$

(b) Let $\delta \leqslant \delta_0$, and $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta$. We start by giving an estimate on $\sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}$. Since the boundary of elements of $\mathcal{M}_{-n}^0$ is contained in $\mathcal{S}_{-l}$, by uniform transversality, each element of $\mathcal{M}_{-n}^0$ is crossed at most one time by $W$. Thus, each element of $\mathcal{M}_0^n$ is crossed at most one time by $T^{-n}W$. Now, since the diameter of elements of $\mathcal{M}_0^n$ is bounded uniformly in $n$ by some constant $C$, there can be no more than $2C\delta^{-1}$ elements of $\mathcal{G}_n^\delta(W)$ in a single element of $\mathcal{M}_0^n$. Thus

$$\sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)} \leqslant 2C\delta^{-1} \sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)} \tag{3.3.3}$$

First, in the case $|W| \geqslant \delta/2$, the estimate

$$\sum_{W_i \in \mathcal{G}_n^\delta(W)} \left(\frac{\log|W|}{\log|W_i|}\right)^{\bar{\gamma}} |e^{S_n g}|_{C^0(W_i)} \leqslant 2C\delta^{-1} 2^{((n \vee n_0)s_0+1)\bar{\gamma}} \sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)},$$

is enough for what we need.

Now, assume that $|W| < \delta/2$. Let $F_1(W)$ denote those $V \in \mathcal{G}_1^\delta(W)$ whose length is at least $\delta/2$. Inductively, define $F_j(W)$, for $2 \leqslant j \leqslant n-1$, to contain those $V \in \mathcal{G}_j^\delta(W)$ whose length is at least $\delta/2$, and such that $T^k V$ is contained in an element of $\mathcal{G}_{j-k}^\delta(W) \smallsetminus F_{j-k}(W)$ for any $1 \leqslant k \leqslant j-1$. Thus $F_j(W)$ contains elements of $\mathcal{G}_j^\delta(W)$ that are "long for the first time" at time $j$.

We group $W_i \in \mathcal{G}_n^\delta(W)$ by its "first long ancestor" as follows. We say $W_i$ has *first long ancestor*[2] $V \in F_j(W)$ for $1 \leqslant j \leqslant n-1$ if $T^{n-j}W_i \subseteq V$. Note that such a $j$ and $V$ are unique for each $W_i$ if they exist. If no such $j$ and $V$ exist, then $W_i$ has been forever short and so must belong to $\mathcal{I}_n^\delta(W)$. Denote by $A_{n-j}(V)$ the set of $W_i \in \mathcal{G}_n^\delta(W)$ corresponding to one $V \in F_j(W)$, that is

$$A_{n-j}(V) \coloneqq \{W_i \in \mathcal{G}_n^\delta(W) \mid T^{n-j}W_i \subset V\}.$$

By construction, we have the relation

$$\mathcal{G}_n^\delta(W) \smallsetminus \left( \bigsqcup_{j=1}^{n-1} \bigsqcup_{V \in F_j(W)} A_{n-j}(V) \right) = \mathcal{I}_n^\delta(W).$$

2. Note that "ancestor" refers to the backwards dynamics mapping $W$ to $W_i$.

Therefore

$$\begin{aligned}
&\sum_{W_i\in\mathcal{G}_n^\delta(W)}\left(\frac{\log|W|}{\log|W_i|}\right)^\gamma|e^{S_ng}|_{C^0(W_i)}\\
&=\sum_{j=1}^{n-1}\sum_{V_l\in F_j(W)}\sum_{W_i\in A_{n-j}(V_l)}\left(\frac{\log|W|}{\log|W_i|}\right)^\gamma|e^{S_ng}|_{C^0(W_i)}+\sum_{W_i\in\mathcal{I}_n(W)}\left(\frac{\log|W|}{\log|W_i|}\right)^\gamma|e^{S_ng}|_{C^0(W_i)}\\
&\leqslant\sum_{j=1}^{n-1}\sum_{V_l\in F_j(W)}\left(\frac{\log|W|}{\log|V_l|}\right)^\gamma|e^{S_jg}|_{C^0(V_l)}\sum_{W_i\in A_{n-j}(V_l)}\left(\frac{\log|V_l|}{\log|W_i|}\right)^\gamma|e^{S_{n-j}g}|_{C^0(W_i)}\\
&\quad+2^{((n\vee n_0)s_0+1)\gamma}(Km+1)^{n/m}e^{n\sup g}\\
&\leqslant 2^{\gamma+1}C\delta^{-1}\sum_{j=1}^{n-1}\sum_{V_l\in F_j(W)}\left(\frac{\log|W|}{\log|V_l|}\right)^\gamma|e^{S_jg}|_{C^0(V_l)}\sum_{A\in\mathcal{M}_0^{n-j}}|e^{S_ng}|_{C^0(A)}\\
&\quad+2^{((n\vee n_0)s_0+1)\gamma}(Km+1)^{n/m}e^{n\sup g}\\
&\leqslant 2^{2\gamma+1}C\delta^{-1}\sum_{j=1}^{n}2^{(j\vee n_0)s_0\gamma}(Km+1)^{j/m}e^{j\sup g}\sum_{A\in\mathcal{M}_0^{n-j}}|e^{S_ng}|_{C^0(A)}.
\end{aligned}$$

where we have applied part (a) from time 1 to time $j$ and the first estimate in part (b) from time $j$ to time $n$, since each $|V_\ell|\geqslant\delta/2$. □

### 3.3.2 Fragmentation Lemmas

Here, given a potential $g$, we introduce the conditions of Small Singular Pressure (SSP.1 and SSP.2) which are crucial for the existence and the statistical properties of the equilibrium states $\mu_g$ that will be constructed in Section 3.6. We prove in Lemma 3.3.3, Corollary 3.3.6 and Lemma 3.3.4 that there exist potentials satisfying simultaneously the conditions SSP.1 and SSP.2. These conditions and their consequences will be used in Section 3.3.3.

In what follows, we will always assume that the potential $g$ is such that $P_*(T,g)-\sup g>s_0\log 2$. Thus, there exists $m$ large enough such that

$$\frac{1}{m}\log(Km+1)<P_*(T,g)-\sup g-s_0\log 2,$$

and we choose $\delta_0=\delta_0(m)$ to be the corresponding length scale from Lemma 3.3.1. Notice that $m$, and therefore also $\delta_0$, depend on $g$.

In order to state the results of this subsection, we give a precise definition of SSP.1. First, we introduce some notations.

Let $L_u^\delta(\mathcal{M}_{-n}^0)$ denote the elements of $\mathcal{M}_{-n}^0$ whose unstable diameter [3] is at least $\delta/3$, for some $\delta\in(0,\delta_0]$. Similarly, let $L_s^\delta(\mathcal{M}_0^n)$ denote the elements of $\mathcal{M}_0^n$ whose stable diameter is at least $\delta/3$. Recall that the boundary of the partition formed by $\mathcal{M}_0^n$ is comprised of stable curves belonging to $\mathcal{S}_n=\cup_{j=0}^n T^{-j}(\mathcal{S}_0)\subset\widehat{\mathcal{W}}^s$.

Define

$$\ell_n^s(g,\delta):=\inf\{\sum_{V\in L_n^\delta(W)}|e^{S_ng}|_{C^0(V)}\mid W\in\widehat{\mathcal{W}}^s,\ \tfrac{\delta}{3}\leqslant|W|\leqslant\delta\},$$

$$\ell_n^u(g,\delta):=\inf\{\sum_{V\in L_n^\delta(W)}|e^{S_n^{-1}g}|_{C^0(V)}\mid W\in\widehat{\mathcal{W}}^u,\ \tfrac{\delta}{3}\leqslant|W|\leqslant\delta\},$$

3. Recall that the unstable diameter of a set is the length of the longest unstable curve contained in that set.

where in the second line $L_n^\delta(W)$, $W \in \widehat{\mathcal{W}}^u$, is defined similarly as in the case $W \in \widehat{\mathcal{W}}^s$, but replacing $T^{-1}$ by $T$ in the definitions, and $S_n^{-1}g := \sum_{i=1}^n g \circ T^{-i} = S_n g \circ T^{-n}$.

**Definition 3.3.2** (SSP.1)**.** *A potential $g$ such that $P(T,g) - \sup g > s_0 \log 2$ is said to have $\varepsilon$-SSP.1 (small singular pressure), for some $\varepsilon > 0$, if*

$$\text{there exist } \delta = \delta(\varepsilon) \in (0, \delta_0] \text{ and } n_1 \in \mathbb{N} \text{ such that} \quad \frac{\sum_{W_i \in L_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}}{\sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}} \geqslant \frac{1-2\varepsilon}{1-\varepsilon}, \quad \forall n \geqslant n_1 \ \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geqslant \delta/3 \quad , \tag{3.3.4}$$

$$\text{the sequences } (e^{n \sup g}\, \ell_n^s(g,\delta)^{-1})_{n \geqslant n_1} \text{ and } (e^{n \sup g}\, \ell_n^u(g,\delta)^{-1})_{n \geqslant n_1} \text{ are summable,} \tag{3.3.5}$$

*and the time reversal of (3.3.4) holds. More precisely, we call time reversal of (3.3.4) the same estimate but replacing $S_n g$ and $W \in \widehat{\mathcal{W}}^s$ with $S_n^{-1}g = \sum_{i=1}^n g \circ T^{-i}$ and $W \in \widehat{\mathcal{W}}^u$. Notice that (3.3.4) (resp. its time reversal) implies that $\ell_n^s(g,\delta)$ (resp. $\ell_n^u(g,\delta)$) is nonzero for all $n \geqslant n_1$.*

*A potential is said to have SSP.1 if it has $\varepsilon$-SSP.1 for some $\varepsilon \leqslant 1/4$.*

The following lemma bootstraps from Lemma 3.3.1 and will be crucial to get the lower bound on the spectral radius:

**Lemma 3.3.3.** *If $g$ is a $(\mathcal{M}_0^1, \alpha)$-Hölder potential such that $P(T,g) - \sup g > s_0 \log 2$ and $\log \Lambda > \sup g - \inf g$, then $g$ satisfies (3.3.4), as well as its time reversal, for all $\varepsilon > 0$.*

*Proof.* Fix $\varepsilon > 0$. Choose $n_1$ so large that $6CC_1^{-1} n_1 (Kn_1 + 1) e^{n_1(\sup g - \inf g - \log \Lambda)} < \varepsilon$, where $C$ is the constant from Lemma 3.2.3 and $C_1$ is such that $|T^{-n}W| > C_1 \Lambda^n |W|$ whenever $W \in \widehat{\mathcal{W}}^s$. Next, choose $\delta > 0$ sufficiently small that if $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta$, then $T^{-n}W$ comprises at most $Kn + 1$ smooth pieces of length at most $\delta_0$ for all $0 \leqslant n \leqslant 2n_1$.

Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geqslant \delta/3$. We shall prove the following equivalent inequality for $n \geqslant n_1$:

$$\frac{\sum_{W_i \in S_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}}{\sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}} \leqslant \frac{\varepsilon}{1-\varepsilon}. \tag{3.3.6}$$

For $n \geqslant n_1$, write $n = kn_1 + l$ for some $0 \leqslant l < n_1$. If $k = 1$, the above inequality is clear since $S_{n_1+l}^\delta(W)$ contains at most $K(n_1 + l) + 1$ components by assumption on $\delta$ and $n_1$, while $|T^{-n_1-l}W| \geqslant C_1 \Lambda^{n_1+l}|W| \geqslant C_1 \Lambda^{n_1+l}\delta/3$. Thus $\mathcal{G}_n^\delta(W)$ must contain at least $C_1\Lambda^{n_1+l}/3$ curves since each has length at most $\delta$. Thus,

$$\frac{\sum_{W_i \in S_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}}{\sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}} \leqslant 3\frac{K(n_1+l)+1}{C_1 \Lambda^{n_1+l}} \frac{e^{(n_1+l)\sup g}}{e^{(n_1+l)\inf g}} \leqslant 6C_1^{-1}(Kn_1+1)e^{n_1(\sup g - \inf g - \log \Lambda)} < \varepsilon,$$

where the last inequality holds by choice of $n_1$.

For $k > 1$, we split $n = kn_1 + l$, $0 \leqslant l < n_1$, into $k-1$ blocks of length $n_1$ and the last block of length $n_1 + l$. For each $V \in \mathcal{G}_n^\delta(W) \smallsetminus \mathcal{I}_n^\delta(W)$, let $j < n$ be the greatest integer such that $T^{n-j}V$ is contained in an element $V_a$ of $L_j^\delta(W)$ and for all $j < i < n$, $T^{n-i}V$ is contained in an element of $S_i^\delta(W)$. We call $V_a$ the *most recent long ancestor* of $V$ and $j$ its *age*. If such a $j$ does not exist, it means that for all $i < n$, $T^{n-i}V$ is short, that is $V \in \mathcal{I}_n^\delta(W)$ and we set $j = 0$ in this case.

We group elements of $S_n^\delta(W)$ by their age in $[0, n_1 - 1]$, $[n_1, 2n_1 - 1], \dots, [(k-2)n_1, (k-1)n_1 - 1]$ and $[(k-1)n_1, n-1]$. In other words, we consider the following decomposition

$$S_n^\delta(W) = \bigsqcup_{q=0}^{k-2}\left(\bigsqcup_{j=qn_1}^{(q+1)n_1-1} \bigsqcup_{V\in L_j^\delta(W)} \mathcal{I}_{n-j}^\delta(V)\right) \sqcup \left(\bigsqcup_{j=(k-1)n_1}^{n-1} \bigsqcup_{V\in L_j^\delta(W)} \mathcal{I}_{n-j}^\delta(V)\right). \tag{3.3.7}$$

We can therefore split the left hand side of (3.3.6) into two manageable parts. For this, we rely on Lemma 3.3.1 for $\gamma = 0$ and the fact that

$$\mathcal{G}_n^\delta(W) \supset \bigsqcup_{V\in L_j^\delta(W)} \mathcal{G}_{n-j}^\delta(V), \quad \forall\, 0 < j < n.$$

Thus, using Lemma 3.2.3, we have

$$\frac{\sum_{q=0}^{k-2}\sum_{j=qn_1}^{(q+1)n_1-1}\sum_{V\in L_j^\delta(W)}\sum_{W_i\in\mathcal{I}_{n-j}^\delta(V)} |e^{S_ng}|_{C^0(W_i)}}{\sum_{W_i\in\mathcal{G}_{kn_1+l}^\delta(W)} |e^{S_ng}|_{C^0(W_i)}}$$

$$\leqslant \sum_{q=0}^{k-2}\sum_{j=qn_1}^{(q+1)n_1-1} \frac{\sum_{V\in L_j^\delta(W)} |e^{S_jg}|_{C^0(V)} \sum_{W_i\in\mathcal{I}_{n-j}^\delta(V)} |e^{S_{n-j}g}|_{C^0(W_i)}}{C^{-1}\sum_{V\in L_j^\delta(W)} |e^{S_jg}|_{C^0(V)} \sum_{W_i\in\mathcal{G}_{n-j}^\delta(V)} e^{(n-j)\inf g}}$$

$$\leqslant \sum_{q=0}^{k-2} 6CC_1^{-1}n_1(Kn_1+1)^{k-q}e^{(k-q)n_1(\sup g-\inf g-\log\Lambda)}$$

$$\leqslant \sum_{q=0}^{k-2}\varepsilon^{k-q} = \sum_{q=2}^{k}\varepsilon^q$$

Similarly, for the second part we have

$$\frac{\sum_{j=(k-1)n_1}^{n-1}\sum_{V\in L_j^\delta(W)}\sum_{W_i\in\mathcal{I}_{n-j}^\delta(V)} |e^{S_ng}|_{C^0(W_i)}}{\sum_{W_i\in\mathcal{G}_{kn_1+l}^\delta(W)} |e^{S_ng}|_{C^0(W_i)}}$$

$$\leqslant \sum_{j=(k-1)n_1}^{n-1} \frac{\sum_{V\in L_j^\delta(W)} |e^{S_jg}|_{C^0(V)} \sum_{W_i\in\mathcal{I}_{n-j}^\delta(V)} |e^{S_{n-j}g}|_{C^0(W_i)}}{C^{-1}\sum_{V\in L_j^\delta(W)} |e^{S_jg}|_{C^0(V)} \sum_{W_i\in\mathcal{G}_{n-j}^\delta(V)} e^{(n-j)\inf g}}$$

$$\leqslant 6CC_1^{-1}n_1(Kn_1+1)e^{n_1(\sup g-\inf g-\log\Lambda)} \leqslant \varepsilon$$

Summing these two estimates, we obtain (3.3.6).

The time reversal is obtained from the same proof by changing the construction of the set $\mathcal{G}_n^\delta(W)$ (and thus $L_n^\delta(W)$, $S_n^\delta(W)$ and $\mathcal{I}_n^\delta(W)$) so that elements of $\mathcal{G}_n^\delta(W)$ are contained in $T^nW$ (instead of $T^{-n}(W)$) for $W \in \widehat{\mathcal{W}}^u$. □

Notice that if $\varepsilon \leqslant 1/4$ and $\delta_1 \leqslant \delta_0$ and $n_1$ are the corresponding $\delta$ and $n_1$ from the $\varepsilon$-SSP.1 condition, then we have for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geqslant \delta_1/3$ and $n \geqslant n_1$

$$\sum_{W_i \in L_n^{\delta_1}(W)} |e^{S_n g}|_{C^0(W_i)} \geqslant \frac{2}{3} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |e^{S_n g}|_{C^0(W_i)}\,, \tag{3.3.8}$$

In particular, since $\mathcal{G}_n^{\delta_1}(W) = L_n^{\delta_1}(W) \sqcup S_n^{\delta_1}(W)$, we also get that

$$\sum_{W_i \in L_n^{\delta_1}(W)} |e^{S_n g}|_{C^0(W_i)} \geqslant 2 \sum_{W_i \in S_n^{\delta_1}(W)} |e^{S_n g}|_{C^0(W_i)}\,.$$

The following lemma will be used to get both lower and upper bounds on the spectral radius via Proposition 3.3.7:

**Lemma 3.3.4.** *Let $g$ be a $(\mathcal{M}_0^1, \alpha)$-Hölder potential such that $P_*(T, g) - \sup g > s_0 \log 2$ and which has SSP.1. Let $\delta_1$ and $n_1$ be the corresponding parameters associated with SSP.1. Then there exist $C_{n_1} > 0$ and $n_2 \geqslant n_1$ such that for all $n \geqslant n_2$,*

$$\begin{aligned} \sum_{A \in L_u^{\delta_1}(\mathcal{M}_{-n}^0)} |e^{S_n^{-1} g}|_{C^0(A)} &\geqslant C_{n_1} \delta_1 \sum_{A \in \mathcal{M}_{-n}^0} |e^{S_n^{-1} g}|_{C^0(A)}, \\ \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)} &\geqslant C_{n_1} \delta_1 \sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)}. \end{aligned} \tag{3.3.9}$$

*Furthermore, if $g$ is a $(\mathcal{M}_0^1, \alpha)$-Hölder potential with $P_*(T, g) - \sup g > s_0 \log 2$ and $\log \Lambda > \sup g - \inf g$, then $g$ has SSP.1.*

*Proof.* We prove the lower bound for $L_s^{\delta_1}(\mathcal{M}_0^n)$. The lower bound for $L_u^{\delta_1}(\mathcal{M}_{-n}^0)$ then follows by time reversal.

First, we need to define sets that will be relevant only here. Let

$$I_s(\mathcal{M}_0^n) \coloneqq \{A \in \mathcal{M}_0^n \mid \operatorname{diam}^s(A) < \delta_1/3\}$$

be the complement of $L_s^{\delta_1}(\mathcal{M}_0^n)$ in $\mathcal{M}_0^n$, and

$$I_s(T^{-j}\mathcal{S}_0) \coloneqq \{\text{unstable curves in } T^{-j}(\mathcal{S}_0) \text{ with length less than } \delta_1/3\}.$$

Define also $L_s(T^{-j}\mathcal{S}_0)$ as the complement of $I_s(T^{-j}\mathcal{S}_0)$ in $\mathcal{G}_j^{\delta_1}(\mathcal{S}_0)$.

We will deduce the claim by estimating the sum of norms of $e^{S_n g}$ over $I_s(\mathcal{M}_0^n)$ by the one over $L_s^{\delta_1}(\mathcal{M}_0^n)$. To do so, we estimate the sum over $I_s(\mathcal{M}_0^n)$ with the sums over $I_s(T^{-j}\mathcal{S}_0)$. Then, using (3.3.4) we estimate the sum over $I_s(T^{-j}\mathcal{S}_0)$ with sums over $L_s(T^{-j}\mathcal{S}_0)$. Finally, we estimate sums over $L_s(T^{-j}\mathcal{S}_0)$ with a sum over $L_s^{\delta_1}(\mathcal{M}_0^n)$.

In order to estimate the sum over $I_s(\mathcal{M}_0^n)$, first remark that if $A \in \mathcal{M}_0^n$ then $\partial A \subset \mathcal{S}_n = \bigcup_{j=0}^n T^{-j}\mathcal{S}_0$. Let $A \in I_s(\mathcal{M}_0^n)$. We distinguish two cases:

(a) For some $1 \leqslant j \leqslant n$, $\partial A$ contains a point of intersection between two curves of $T^{-j}\mathcal{S}_0$. Since such intersection point is the image by $T^{-j+1}$ of an intersection point between curves of $T^{-1}\mathcal{S}_0$, which are finite, and thank to the linear complexity (3.3.1), we get that there are at most $K_2 n$ elements of $I_s(\mathcal{M}_0^n)$ in this case.

(b) $\partial A$ only contains intersection points between curves belonging to $T^{-j}\mathcal{S}_0$ for different $j$. Let $j_A$ be the maximal $1 \leqslant j \leqslant n$ such that $A \cap T^{-j}\mathcal{S}_0 \neq \emptyset$, and $\gamma \in T^{-j_A}\mathcal{S}_0$ such that $\gamma \cap A \neq \emptyset$. Notice that $\gamma$ must intersect other curves from $\partial A$. These curves belong to

$T^{-j}\mathcal{S}_0$ for some $j < j_A$. Applying $T^j$, it appears that $\gamma$ must terminate at these intersection points, and thus $\gamma \subset \partial A$. Since $\gamma$ is a stable curve, $\gamma$ belongs to $I_s(T^{-j_A}\mathcal{S}_0)$ by assumption on $A$. Finally, such a curve $\gamma$ belong to at most 2 elements of $I_s(\mathcal{M}_0^n)$.

Therefore

$$\sum_{A \in I_s(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)} \leqslant K_2 n e^{n \sup g} + C \sum_{j=1}^{n} \sum_{W \in I_s(T^{-j}\mathcal{S}_0)} |e^{S_n g}|_{C^0_+(W)} + |e^{S_n g}|_{C^0_-(W)}, \tag{3.3.10}$$

where we have extended $e^{S_n g}$ by Hölder continuity to $W$ from both sides – and noted $|\cdot|_{C^0_+(W)}$ and $|\cdot|_{C^0_-(W)}$ the corresponding norms – and $C$ is the constant from Lemma 3.2.3.

In order to use (3.3.4), we decompose $\mathcal{S}_0 = \bigsqcup_{i=1}^{l_0} U_i$ where each $U_i$ is a connected curve such that $\frac{\delta_1}{3} \leqslant |T^{-1}U_i| \leqslant \delta_1$. But first we need to compare the sum indexed by $I_s(T^{-j}\mathcal{S}_0)$ with the one indexed by $I_s(\mathcal{G}^{\delta_1}_{j-1}(U_i))$. Let $W \in I_s(T^{-j}\mathcal{S}_0)$. Thus, each $W \cap T^{-j}U_i$ is a single maximal smooth component of length less than $\delta_1/3$. In other words, $W \cap T^{-j}U_i \in I_s(\mathcal{G}^{\delta_1}_{j-1}(U_i))$. Therefore

$$\sum_{W \in I_s(T^{-j}\mathcal{S}_0)} |e^{S_n g}|_{C^0_\pm(W)} \leqslant \sum_{i=1}^{l_0} \sum_{W \in I_s(\mathcal{G}^{\delta_1}_{j-1}(T^{-1}U_i))} |e^{S_n g}|_{C^0_\pm(W)}. \tag{3.3.11}$$

Now, using SSP.1 (3.3.4), in the case $j > n_1$, we get that

$$\sum_{W \in I_s(\mathcal{G}^{\delta_1}_{j-1}(T^{-1}U_i))} |e^{S_n g}|_{C^0_\pm(W)} \leqslant \frac{1}{2} e^{(n-j+1)\sup g} \sum_{W \in L_s(\mathcal{G}^{\delta_1}_{j-1}(T^{-1}U_i))} |e^{S_{j-1} g}|_{C^0_\pm(W)}. \tag{3.3.12}$$

In order to estimate this last sum with the sum indexed by $L_s(\mathcal{G}^{\delta_1}_{n-1}(T^{-1}U_i))$, notice that

$$L_s(\mathcal{G}^{\delta_1}_{n-1}(T^{-1}U_i)) \supset \bigsqcup_{V \in L_s(\mathcal{G}^{\delta_1}_{j-1}(T^{-1}U_i))} L_s(\mathcal{G}^{\delta_1}_{n-j}(V)).$$

Thus

$$\begin{aligned}
\sum_{W \in L_s(\mathcal{G}^{\delta_1}_{n-1}(T^{-1}U_i))} |e^{S_n g}|_{C^0_\pm(W)} &\geqslant \sum_{W \in L_s(\mathcal{G}^{\delta_1}_{j-1}(T^{-1}U_i))} \sum_{V \in L_s(\mathcal{G}^{\delta_1}_{n-j}(W))} |e^{S_{n-j} g + S_j g \circ T^{n-j}}|_{C^0_\pm(V)} \\
&\geqslant C^{-2} e^{\inf g} \sum_{W \in L_s(\mathcal{G}^{\delta_1}_{j-1}(T^{-1}U_i))} |e^{S_{j-1} g}|_{C^0_\pm(W)} \sum_{V \in L_s(\mathcal{G}^{\delta_1}_{n-j}(W))} |e^{S_{n-j} g}|_{C^0_\pm(V)} \\
&\geqslant C^{-2} e^{\inf g}\, \ell^s_{n-j}(g, \delta_1) \sum_{W \in L_s(\mathcal{G}^{\delta_1}_{j-1}(T^{-1}U_i))} |e^{S_{j-1} g}|_{C^0_\pm(W)},
\end{aligned}$$

where we used Lemma 3.2.3 for the second inequality, and the definition of $\ell^s_{n-j}(g, \delta_1)$ for the third inequality. Notice however that (3.3.4) ensures that $\ell^s_{n-j}(g, \delta_1) \neq 0$ only for $n - j \geqslant n_1$. We will treat these troublesome $j$ afterwards. Assume for now that $n_1 \leqslant j \leqslant n - n_1$. Combining the above lower bound with (3.3.11) and (3.3.12), we get

$$\sum_{W \in I_s(T^{-j}\mathcal{S}_0)} |e^{S_n g}|_{C^0_\pm(W)} \leqslant \bar{C}\, e^{(n-j)\sup g} \ell^s_{n-j}(g, \delta_1)^{-1} \sum_{W \in L_s(T^{-n}\mathcal{S}_0)} |e^{S_n g}|_{C^0_\pm(W)}, \tag{3.3.13}$$

where we used that $\bigsqcup_{i=1}^{l_0} L_s(\mathcal{G}^{\delta_1}_{j-1}(T^{-1}U_i)) \subset L_s(T^{-j}\mathcal{S}_0)$ – which is true if we choose the $\delta_1$-scaling $\mathcal{G}_1(T^{-j}\mathcal{S}_0)$ to be adapted with the decomposition $\mathcal{S}_0 = \bigsqcup_i U_i$.

Now, if $n-n_1 \leqslant j \leqslant n$, then we obtain from similar computations

$$\sum_{W\in I_s(T^{-j}\mathcal{S}_0)} |e^{S_n g}|_{C^0_\pm(W)} \leqslant \frac{1}{2}C^2 e^{\inf g} e^{(n-j+1)\sup g} \ell^s_{n_1}(g,\delta_1)^{-1} \sum_{W\in L_s(T^{-j-n_1}\mathcal{S}_0)} |e^{S_{n_1+j} g}|_{C^0_\pm(W)} \tag{3.3.14}$$

Finally, we estimate the sum over $L_s(T^{-n}\mathcal{S}_0)$ with the sum over $L_s^{\delta_1}(\mathcal{M}_0^n)$. To do so, we use similar arguments than for the estimate (3.3.10). Let $W \in L_s(T^{-n}\mathcal{S}_0)$. We distinguish the two following cases:
(a) $W$ intersects another curve from $T^{-n}\mathcal{S}_0$. There are at most $2K_2$ elements of $L_s(T^{-n}\mathcal{S}_0)$ in this case,
(b) $W$ does not intersect other curves from $T^{-n}\mathcal{S}_0$. In that case, $W$ must be contained in the boundary of an element of $\mathcal{M}_0^n$, and thus an element of $L_s^{\delta_1}(\mathcal{M}_0^n)$. Now, there are at most $2C\delta_1^{-1}$ elements of $L_s(T^{-n}\mathcal{S}_0)$ in the boundary of a single element of $L_s^{\delta_1}(\mathcal{M}_0^n)$, where $C$ is a large enough constant depending only on the billiard table.

Thus

$$\sum_{W\in L_s(T^{-n}\mathcal{S}_0)} |e^{S_n g}|_{C^0_\pm(W)} \leqslant 2K_2 e^{n\sup g} + C\delta_1^{-1} \sum_{A\in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)}. \tag{3.3.15}$$

Similarly, for all $n-n_1 \leqslant j \leqslant n$,

$$\sum_{W\in L_s(T^{-n_1-j}\mathcal{S}_0)} |e^{S_{n_1+j} g}|_{C^0_\pm(W)} \leqslant 2K_2 e^{(n_1+j)\sup g} + C\delta_1^{-1} \sum_{A\in L_s^{\delta_1}(\mathcal{M}_0^{n_1+j})} |e^{S_{n_1+j} g}|_{C^0(A)}. \tag{3.3.16}$$

Putting together (3.3.10), (3.3.13) and (3.3.15), as well as (3.3.14) and (3.3.16), we get

$$\begin{aligned}
\sum_{A\in I_s(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)} &\leqslant K_2 n e^{n\sup g} + C\sum_{j=1}^{n_1-1} \sum_{W\in I_s(T^{-j}\mathcal{S}_0)} |e^{S_n g}|_{C^0_+(W)} + |e^{S_n g}|_{C^0_-(W)} \\
&\quad + C\sum_{j=n_1}^{n-n_1} \sum_{W\in I_s(T^{-j}\mathcal{S}_0)} |e^{S_n g}|_{C^0_+(W)} + |e^{S_n g}|_{C^0_-(W)} \\
&\quad + C\sum_{j=n-n_1+1}^{n} \sum_{W\in I_s(T^{-j}\mathcal{S}_0)} |e^{S_n g}|_{C^0_+(W)} + |e^{S_n g}|_{C^0_-(W)} \\
&\leqslant (K_2 n + \bar{C}_{n_1}) e^{n\sup g} + \bar{C}\sum_{j=n_1}^{n-n_1} e^{j\sup g}\, \ell^s_j(g,\delta_1)^{-1} \sum_{W\in L_s(T^{-n}\mathcal{S}_0)} |e^{S_n g}|_{C^0_+(W)} + |e^{S_n g}|_{C^0_-(W)} \\
&\quad + C\sum_{j=n-n_1+1}^{n} e^{(n-j)\sup g} \ell^s_{n_1}(g,\delta_1)^{-1} \sum_{W\in L_s(T^{-j-n_1}\mathcal{S}_0)} |e^{S_{j+n_1} g}|_{C^0_+(W)} + |e^{S_{j+n_1} g}|_{C^0_-(W)} \\
&\leqslant (K_2 n + \bar{C}_{n_1}) e^{n\sup g} + \tilde{C}\left( 2K_2 e^{n\sup g} + C\delta_1^{-1} \sum_{A\in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)} \right) \\
&\quad + \sum_{j=n-n_1+1}^{n} C'_{n_1} C'_g \left( 2K_2 e^{(n_1+j)\sup g} + C\delta_1^{-1} \sum_{A\in L_s^{\delta_1}(\mathcal{M}_0^{j+n_1})} |e^{S_{j+n_1} g}|_{C^0(A)} \right),
\end{aligned}$$

where in the last inequality we used (3.3.5) and the fact that $n-n_1+1 \leqslant j \leqslant n$ is equivalent to $0 \leqslant n-j \leqslant n_1-1$, that is, in the second sum over $j$ after the second inequality symbol, the $e^{(n-j)\sup g}$ are uniformly bounded (by $C'_g$).

We now relate the sum over $L_s^{\delta_1}(\mathcal{M}_0^{j+n_1})$ to the sum over $L_s^{\delta_1}(\mathcal{M}_0^n)$. To do so, notice that if $A \in L_s^{\delta_1}(\mathcal{M}_0^n)$, then it contains at most $B^{j+n_1-n}$ elements of $L_s^{\delta_1}(\mathcal{M}_0^{j+n_1})$, where $B = |\mathcal{P}|$. On the other hand, an element $A' \in L_s^{\delta_1}(\mathcal{M}_0^{j+n_1})$ is contained in exactly one element of $L_s^{\delta_1}(\mathcal{M}_0^n)$. Thus

$$\begin{aligned}
\sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^{j+n_1})} |e^{S_{j+n_1}g}|_{C^0(A)} &= \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^{j+n_1})} \sum_{\substack{A' \in L_s^{\delta_1}(\mathcal{M}_0^n) \\ A \subset A'}} |e^{S_{j+n_1}g}|_{C^0(A)} \\
&\leqslant \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^{j+n_1})} \sum_{\substack{A' \in L_s^{\delta_1}(\mathcal{M}_0^n) \\ A \subset A'}} e^{n_1 \sup g} |e^{S_j g}|_{C^0(A')} \\
&\leqslant \sum_{A' \in L_s^{\delta_1}(\mathcal{M}_0^n)} \sum_{\substack{A \in L_s^{\delta_1}(\mathcal{M}_0^{j+n_1}) \\ A \subset A'}} e^{n_1 \sup g} |e^{S_j g}|_{C^0(A')} \\
&\leqslant B^{j+n_1-n} e^{n_1 \sup g} \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_j g}|_{C^0(A)},
\end{aligned}$$

and therefore,

$$\begin{aligned}
\sum_{j=n-n_1+1}^{n} \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^{j+n_0})} |e^{S_{j+n_1}g}|_{C^0(A)} &\leqslant \sum_{j=n-n_1+1}^{n} B^{j+n_1-n} e^{n_1 \sup g} \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_j g}|_{C^0(A)} \\
&\leqslant \sum_{j=n-n_1+1}^{n} B^{j+n_1-n} e^{n_1 \sup g} e^{(n-j)\inf g} \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)} \\
&\leqslant \tilde{C}_{n_1} \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)}.
\end{aligned}$$

Using this last estimate, we obtain

$$\begin{aligned}
\sum_{A \in I_s(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)} &\leqslant (K_2 n + \bar{C}_{n_1} + C'_{n_1} C'_g K_2) e^{n \sup g} \\
&\quad + (\tilde{C}C + C'_{n_1} C'_g \tilde{C}_{n_1}) \delta_1^{-1} \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)}, \\
&\leqslant C_1 e^{n \sup g} + C_2 \delta_1^{-1} \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)},
\end{aligned}$$

where $\tilde{C}$ is a constant coming from the summability assumption (3.3.5), and $\bar{C}_{n_1}$ depends only on $n_1$ and $g$.

Finally, since $I_s(\mathcal{M}_0^n) \sqcup L_s^{\delta_1}(\mathcal{M}_0^n) = \mathcal{M}_0^n$, we get that

$$\sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^n)} |e^{S_n g}|_{C^0(A)} \geqslant \frac{\sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)} - C_1 e^{n \sup g}}{1 + C_2 \delta_1^{-1}}.$$

Since $\lim_{n\to+\infty}\frac{1}{n}\log\sum_{A\in\mathcal{M}_0^n}|e^{S_ng}|_{C^0(A)} = P_*(T,g)$ and by the assumption $P_*(T,g) > \sup g$, there is an integer $n_2$ such that for all $n \geqslant n_2$,

$$\sum_{A\in\mathcal{M}_0^n}|e^{S_ng}|_{C^0(A)} - C_1e^{n\sup g} \geqslant \frac{1}{2}\sum_{A\in\mathcal{M}_0^n}|e^{S_ng}|_{C^0(A)}.$$

Thus, there exists $C_{n_1} > 0$ such that for all $n \geqslant n_2$ (3.3.9) holds.

We now prove the second part of Lemma 3.3.4. Assume that $g$ is a $(\mathcal{M}_0^1,\alpha)$-Hölder potential with $P_*(T,g) - \sup g > s_0\log 2$ and $\log\Lambda > \sup g - \inf g$. From the convexity of the topological pressure (Theorem 3.2.1), we get that $t\mapsto P_*(T,tg)$ is a convex function. Thus, the map $t\mapsto P_*(T,t(g-\sup g)) = P_*(T,tg) - t\sup g$ is continuous on $[0,1]$. Since for all $s<t$ we have

$$\sum_{A\in\mathcal{M}_0^n}|e^{S_nt(g-\sup g)}|_{C^0(A)} \leqslant e^{n(t-s)\sup(g-\sup g)}\sum_{A\in\mathcal{M}_0^n}|e^{S_ns(g-\sup g)}|_{C^0(A)} = \sum_{A\in\mathcal{M}_0^n}|e^{S_ns(g-\sup g)}|_{C^0(A)},$$

the map is nonincreasing. Thus

$$P_*(T,g) - \sup g = P_*(T,g-\sup g) \leqslant P_*(T,0) = h_*,$$

where $h_*$ is the topological entropy from [BD20]. Therefore we have $h_* > s_0\log 2$ and estimates from [BD20] can be used. For all $W\in\widehat{\mathcal{W}}^s$ with $\delta_1 \geqslant |W| \geqslant \delta_1/3$ and all $n\geqslant n_1$,

$$\begin{aligned}\sum_{V\in L_n^{\delta_1}(W)}|e^{S_ng}|_{C^0(V)} &\geqslant e^{n\inf g}\#L_n^{\delta_1}(W) \geqslant \frac{2}{3}e^{n\inf g}\#\mathcal{G}_n^{\delta_1}(W) \geqslant \frac{2}{3}c_0e^{n\inf g}\#\mathcal{M}_0^n\\ &\geqslant \frac{2}{3}c_0e^{n(\inf g + P_*(T,0))}\end{aligned}$$

where we used [BD20, Lemma 5.2] for the second inequality, and Propositions 4.6 and 5.5 from [BD20] in the third inequality [4].

Thus we get that $\ell_n^s(g,\delta_1) \geqslant \frac{2}{3}c_0e^{n(\inf g+P_*(T,0))}$. Since [5] $P_*(T,0) = h_* \geqslant \log\Lambda$, we then get the summability of the sequence $(e^{n\sup g}\ell_n^s(g,\delta_1)^{-1})_{n\geqslant n_1}$. The summability of $e^{n\sup g}\ell_n^u(g,\delta_1)^{-1}$ is obtained similarly by considering lower bounds on $\#L_u^{\delta_1}(W)$, also given in [BD20]. □

We now introduce the precise definition of SSP.2:

**Definition 3.3.5** (SSP.2)**.** *A potential $g$ is said to have $\varepsilon$-SSP.2 if it has $\varepsilon$-SSP.1, if there exists $\bar{n}_1 : (0,+\infty)\to\mathbb{N}$ such that*

$$\frac{\sum_{W_i\in L_n^\delta(W)}|e^{S_ng}|_{C^0(W_i)}}{\sum_{W_i\in\mathcal{G}_n^\delta(W)}|e^{S_ng}|_{C^0(W_i)}} \geqslant \frac{1-3\varepsilon}{1-\varepsilon},\quad \forall W\in\widehat{\mathcal{W}}^s,\ \forall n\geqslant\bar{n}_1(|W|), \tag{3.3.17}$$

*and if the time reversal [6] of (3.3.17) holds, where $\delta$ is the corresponding constant from $\varepsilon$-SSP.1. A potential is said to have SSP.2 if it has $\varepsilon$-SSP.2 for some $\varepsilon\leqslant 1/4$.*

4. We can choose the scale $\delta_1$ from [BD20] to agree with the one here. The constant $c_0$ comes from [BD20, Proposition 5.5] and depends on $\delta_1$.

5. $\log\Lambda$ is a lower bound on the unstable Lyapunov exponent of $T$. Integrating against $\mu_{\mathrm{SRB}}$ gives the desired inequality.

6. As for (3.3.4), we call time reversal of (3.3.17) the same estimate but with $S_ng$ and $W\in\widehat{\mathcal{W}}^s$ replaced by $S_n^{-1}g$ and $W\in\widehat{\mathcal{W}}^u$.

**Corollary 3.3.6.** *If $g$ is a $(\mathcal{M}_0^1, \alpha)$-Hölder potential such that $P(T,g) - \sup g > s_0 \log 2$ and $\log \Lambda > \sup g - \inf g$, then there exists $C_2 > 0$ such that $g$ has $\varepsilon$-SSP.2 for all $\varepsilon > 0$ and $\bar{n}_1(|W|) = C_2 n_1 \frac{|\log(|W|/\delta)|}{|\log \varepsilon|}$, where $\delta$ and $n_1$ are the corresponding constants from Lemma 3.3.3.*

*Proof.* From the Lemmas 3.3.3 and 3.3.4, such a potential has SSP.1. We thus only prove (3.3.17).

The proof is essentially the same as the one for Lemma 3.3.3, except that for curves shorter than $\delta/3$ one must wait $n \lesssim |\log(|W|/\delta)|$ for at least one component of $\mathcal{G}_n^\delta(W)$ to belong to $L_n^\delta(W)$.

More precisely, fix $\varepsilon > 0$ and the corresponding $\delta$ and $n_1$ from Lemma 3.3.3. Let $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta/3$ and take $n > n_1$. Decomposing $\mathcal{G}_n^\delta(W)$ and $S_n^\delta(W)$ as in the proof of Lemma 3.3.3, we estimate the second part as before. For the first part, we have to split the sum between $\mathcal{I}_n^\delta(W)$ and the rest, which is estimated as before.

For the first part, concerning $\mathcal{I}_n^\delta(W)$, for $\delta$ sufficiently small, notice that since the flow is continuous, either $\#\mathcal{G}_l^\delta(W) \leqslant Kl + 1$ by (3.3.1) or at least one element of $\mathcal{G}_l^\delta(W)$ has length at least $\delta/3$. Let $n_2$ denote the first iterate $l$ at which $\mathcal{G}_l^\delta(W)$ contains at least one element of length more than $\delta/3$. By the complexity estimate (3.3.1) and the fact that $|T^{-n_2}W| \geqslant C_1 \Lambda^{n_2} |W|$ by hyperbolicity of $T$, there exists $\overline{C}_2 > 0$, independent of $W \in \widehat{\mathcal{W}}^s$, such that $n_2 \leqslant \overline{C}_2 |\log(|W|/\delta)|$.

Now, for $n \geqslant n_2$,

$$\begin{aligned}\sum_{W_i \in \mathcal{I}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)} &\leqslant \sum_{W' \in \mathcal{G}_{n_2}^\delta(W)} |e^{S_{n_2} g}|_{C^0(W')} \sum_{W_i \in \mathcal{I}_{n-n_2}^\delta(W')} |e^{S_{n-n_2} g}|_{C^0(W_i)} \\ &\leqslant K(Kn_2+1) e^{n_2 \sup g} \times 2(Kn_1+1)^{\frac{n-n_2}{n_1}} e^{(n-n_2)\sup g}\end{aligned}$$

and by hyperbolicity and Lemma 3.2.3,

$$\begin{aligned}\sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)} &\geqslant C^{-1} |e^{S_{n_2} g}|_{C^0(W')} \sum_{W_i \in \mathcal{G}_{n-n_2}^\delta(W')} e^{(n-n_2)\inf g} \\ &\geqslant \frac{1}{3} C_1 C^{-1} e^{n_2 \inf g} e^{(n-n_2)(\inf g + \log \Lambda)}\end{aligned}$$

where $W' \in \mathcal{G}_{n_2}^\delta(W)$ is such that $|W'| > \delta/3$. Therefore,

$$\begin{aligned}\frac{\sum_{W_i \in \mathcal{I}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}}{\sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{S_n g}|_{C^0(W_i)}} &\leqslant 6 C_1^{-1} C e^{n_2(\sup g - \inf g)} K(Kn_2+1)(Kn_1+1)^{\frac{n-n_2}{n_1}} e^{(n-n_2)(\sup g - \inf g - \log \Lambda)} \\ &\leqslant 2 c_0^{-1} C^2 e^{n_2(\sup g - \inf g)} K(Kn_2+1) \varepsilon^{n/n_1}.\end{aligned}$$

Since $n_2 \leqslant \overline{C}_2 |\log(|W|/\delta)|$, we can bound this expression by $\varepsilon$ by choosing some $C_2 > 0$ and $n$ large enough so that $n/n_1 \geqslant C_2 \frac{\log(|W|/\delta)}{\log \varepsilon}$. For such $n$, the left hand side of (3.3.6) is bounded by $\varepsilon + \frac{\varepsilon}{1-\varepsilon} \leqslant \frac{2\varepsilon}{1-\varepsilon}$, which completes the proof of the corollary.

As usual, the time reversal of (3.3.17) is obtained by performing the same proof, but with the time reversal counterpart of $\mathcal{G}_n^\delta(W)$, for unstable curves $W$. $\square$

### 3.3.3 Exact Exponential Growth of Thermodynamic Sums – Cantor Rectangles

It follows from the submultiplicativity in the characterisation of $P_*(T,g)$ that

$$e^{nP_*(T,g)} \leqslant e^{-\inf g} \sum_{A\in\mathcal{M}_0^n} \sup_{x\in A} e^{(S_n g)(x)}$$

for all $n$. In this subsection, we shall prove a supermultiplicativity statement (Lemma 3.3.9) from which we deduce the upper bound for $\sum_{A\in\mathcal{M}_0^n} \sup_{x\in A} e^{(S_n g)(x)}$ in Proposition 3.3.10 giving the upper bound in Proposition 3.5.1, and ultimately the upper bound on the spectral radius of $\mathcal{L}_g$ on $\mathcal{B}$.

The following key estimate is a lower bound on the weighted rate of growth of stable curves having a certain length. The proof will crucially use the fact that the SRB measure is mixing in order to bootstrap from SSP.1.

**Proposition 3.3.7.** *Let $g$ be a $(\mathcal{M}_0^1,\alpha)$-Hölder potential with $P_*(T,g)-\sup g > s_0\log 2$ and which has SSP.1. Let $\delta_1$ be the value of $\delta$ from the condition SSP.1. Then there exists $c_0>0$ such that for all $W\in\widehat{\mathcal{W}}^s$ with $|W|\geqslant \delta_1/3$ and $n\geqslant 1$, we have*

$$\sum_{W_i\in\mathcal{G}_n^{\delta_0}(W)} |e^{S_n g}|_{C^0(A)} \geqslant c_0 \sum_{A\in\mathcal{M}_{-n}^0} |e^{S_n^{-1} g}|_{C^0(A)}.$$

*The constant $c_0$ depends on $\delta_1$.*

The proof relies crucially on the notion of *Cantor rectangles.* We introduce this notion as in [BD20, Definition 5.7]. Let $W^s(x)$ and $W^u(x)$ denote the maximal smooth components of the local stable and unstable manifolds of $x\in M$.

**Definition 3.3.8.** *A solid rectangle $D$ in $M$ is a closed connected set whose boundary comprises precisely four nontrivial curves: two (segments of) stable manifolds and two (segments of) unstable manifolds. Given a solid rectangle $D$, the (locally maximal) Cantor rectangle $R$ in $D$ is formed by taking the points in $D$ whose local stable and unstable manifolds completely cross $D$. Cantor rectangles have a natural product structure: for any $x$, $y\in R$, then $W^s(x)\cap W^u(y)\in R$. In [CM06, Section 7.11], Cantor rectangles are proved to be closed, and thus contain their outer boundaries, which are contained in the boundary of $D$. With a slight abuse, we will call these pairs of stable and unstable manifolds the stable and unstable boundaries of $R$. In this case, we denote $D$ by $D(R)$ to emphasize that it is the smallest solid rectangle containing $R$.*

*Proof of Proposition 3.3.7.* Using [CM06, Lemma 7.87], we may cover $M$ by Cantor rectangles $R_1,\dots,R_k$ satisfying

$$\inf_{x\in R_i} \frac{m_{W^u}(W^u(x)\cap R_i)}{m_{W^u}(W^u(x)\cap D(R_i))} \geqslant 0.9\,, \quad \forall 1\leqslant i\leqslant k, \tag{3.3.18}$$

whose stable and unstable boundaries have lengths at most $\frac{1}{10}\delta_1$, with the property that any stable curve of length at least $\delta_1/3$ *properly crosses* at least one of them. A stable curve $W\in\widehat{\mathcal{W}}^s$ is said to properly cross $R$ if $W$ crosses both unstable sides of $R$, $W$ does not cross any stable manifolds $W^s(x)\cap D(R)$ for $x\in R$, and the point $W\cap W^u(x)$ subdivides the curve $W^u(x)\cap D(R)$ in a ratio between 0.1 and 0.9 (i.e. $W$ does not come to close to either stable boundary of $R$). The cardinality $k$ is fixed, depending only on $\delta_1$.

Recall that $L_u^{\delta_1}(\mathcal{M}_{-n}^0)$ denotes the elements of $\mathcal{M}_{-n}^0$ whose unstable diameter is longer than $\delta_1/3$. We claim that for all $n \in \mathbb{N}$, at least one $R_i$ is fully crossed in the unstable direction by each element in a subset $\tilde{L}$ of $\mathcal{M}_{-n}^0$ such that

$$\sum_{A\in\tilde{L}} |e^{S_n^{-1}g}|_{C^0(A)} \geqslant \frac{1}{k} \sum_{A\in L_u^{\delta_1}(\mathcal{M}_{-n}^0)} |e^{S_n^{-1}g}|_{C^0(A)}. \tag{3.3.19}$$

Notice that if $A \in \mathcal{M}_{-n}^0$, then $\partial A$ is comprised of unstable curves belonging to $\cup_{i=1}^n T^i\mathcal{S}_0$, and possibly $\mathcal{S}_0$. By definition of unstable manifolds, $T^i\mathcal{S}_0$ cannot intersect the unstable boundaries of the $R_i$; thus if $A \cap R_i \neq \emptyset$, then either $\partial A$ terminates inside $R_i$ or $A$ fully crosses $R_i$. Thus elements of $L_u^{\delta_1}(\mathcal{M}_{-n}^0)$ fully cross at least one $R_i$ and so at least one $R_i$ must be fully crossed by *a large fraction* $\tilde{L}$ of $L_u^{\delta_1}(\mathcal{M}_{-n}^0)$ in the sense of (3.3.19), proving the claim.

For each $n \in \mathbb{N}$, denote by $i_n$ the index of a rectangle $R_{i_n}$ which is fully crossed by a large enough subset $\tilde{L}_n$ of $L_u(\mathcal{M}_{-n}^0)$, in the sense of (3.3.19).

Fix $\delta_* \in (0, \delta_1/10)$ and for $i = 1, ...k$, choose a "high density" subset $R_i^* \subset R_i$ satisfying the following conditions: $R_i^*$ has a non-zero Lebesgue measure, and for any unstable manifold $W^u$ such that $W^u \cap R_i^* \neq \emptyset$ and $|W^u| < \delta_*$, we have $\frac{m_{W^u}(W^u\cap R_i^*)}{|W^u|} \geqslant 0.9$. (Such a $\delta_*$ and $R_i^*$ exist due to the fact that $m_{W^u}$-almost every $y \in R_i$ is a Lebesgue density point of the set $W^u(y) \cap R_i$ and the unstable foliation is absolutely continuous with respect to $\mu_{\text{SRB}}$ or, equivalently, Lebesgue.)

Due to the mixing property of $\mu_{\text{SRB}}$ and the finiteness of the number of rectangles $R_i$, there exist $\varepsilon > 0$ and $n_3 \in \mathbb{N}$ such that for all $1 \leqslant i,j \leqslant k$ and all $n \geqslant n_3$, $\mu_{\text{SRB}}(R_i^* \cap T^{-n}R_j) \geqslant \varepsilon$. If necessary, we increase $n_3$ so that the unstable diameter of the set $T^{-n}R_i$ is less than $\delta_*$ for each $i$, and $n \geqslant n_3$.

Now let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geqslant \delta_1/3$ be arbitrary. Let $R_j$ be a Cantor rectangle that is properly crossed by $W$. Let $n \in \mathbb{N}$ and let $i_n$ be as above. By mixing, $\mu_{\text{SRB}}(R_{i_n}^* \cap T^{-n_3}R_j) \geqslant \varepsilon$. By [CM06, Lemma 7.90], there is a component of $T^{-n_3}W$ that fully crosses $R_{i_n}^*$ in the stable direction. Call this component $V \in \mathcal{G}_{n_3}^{\delta_0}(W)$. Thus

$$\sum_{W_i\in\mathcal{G}_n^{\delta_0}(V)} |e^{S_ng}|_{C^0(W_i)} = \sum_{W_i\in\mathcal{G}_n^{\delta_0}(V)} |e^{S_n^{-1}g}|_{C^0(T^nW_i)} \geqslant \sum_{A\in\tilde{L}_n} \inf_A |e^{S_n^{-1}g}| \geqslant \frac{1}{C_g} \sum_{A\in\tilde{L}_n} \sup_A |e^{S_n^{-1}g}|$$
$$\geqslant \frac{1}{k} \sum_{A\in L_u^{\delta_1}(\mathcal{M}_{-n}^0)} |e^{S_n^{-1}g}|_{C^0(A)}.$$

We now have to relate the lhs to the analogous quantity where $V$ is replace by $W$.

$$\sum_{W_i\in\mathcal{G}_n^{\delta_0}(W)}|e^{S_ng}|_{C^0(W_i)} = \sum_{V_j\in\mathcal{G}_{n+n_3}^{\delta_0}(W)}\sum_{\substack{W_i\in\mathcal{G}_n^{\delta_0}(W)\\ T^{n_3}V_j\subset W_i}}\frac{|e^{S_ng}|_{C^0(W_i)}}{\#\{V_j\in\mathcal{G}_{n+n_3}^{\delta_0}(W)\mid T^{n_3}V_j\subset W_i\}}$$

$$\geqslant \sum_{V_j\in\mathcal{G}_{n+n_3}^{\delta_0}(W)}|e^{S_ng\circ T^{n_3}}|_{C^0(V_j)}\sum_{\substack{W_i\in\mathcal{G}_n^{\delta_0}(W)\\ T^{n_3}V_j\subset W_i}}\frac{1}{\#\{V_j\in\mathcal{G}_{n+n_3}^{\delta_0}(W)\mid T^{n_3}V_j\subset W_i\}}$$

$$\geqslant \frac{C\delta_0}{\#\mathcal{M}_0^{n_3}}e^{-n_3\sup g}\sum_{V_j\in\mathcal{G}_{n+n_3}^{\delta_0}(W)}|e^{S_{n+n_3}g}|_{C^0(V_j)} \geqslant \frac{C\delta_0}{\#\mathcal{M}_0^{n_3}}e^{-n_3\sup g}\sum_{V_j\in\mathcal{G}_n^{\delta_0}(V)}|e^{S_{n+n_3}g}|_{C^0(V_j)}$$

$$\geqslant \frac{C\delta_0}{\#\mathcal{M}_0^{n_3}}e^{-n_3(\sup g-\inf g)}\sum_{V_j\in\mathcal{G}_n^{\delta_0}(V)}|e^{S_ng}|_{C^0(V_j)}$$

$$\geqslant \frac{1}{kC_g}\frac{C\delta_0}{\#\mathcal{M}_0^{n_3}}e^{-n_3(\sup g-\inf g)}\sum_{A\in L_u^{\delta_1}(\mathcal{M}_{-n}^0)}|e^{S_n^{-1}g}|_{C^0(A)}$$

$$\geqslant C_{n_1}\delta_1\frac{1}{kC_g}\frac{C\delta_0}{\#\mathcal{M}_0^{n_3}}e^{-n_3(\sup g-\inf g)}\sum_{A\in\mathcal{M}_{-n}^0}|e^{S_n^{-1}g}|_{C^0(A)},$$

for all $n\geqslant\max\{n_2,n_3\}$, where we used Lemma 3.3.4 for the last inequality. Thus the proposition holds for all $n\geqslant\max\{n_2,n_3\}$. It extends to all $n\in\mathbb{N}$ since there are finitely many values of $n$ to correct for. □

**Lemma 3.3.9** (Supermultiplicativity). *There exists a constant $c_1$ such that for all $n\in\mathbb{N}$, and all $0<j<n$, we have*

$$\sum_{A\in\mathcal{M}_0^n}|e^{S_ng}|_{C^0(A)}\geqslant c_1\sum_{A\in\mathcal{M}_0^{n-j}}|e^{S_{n-j}g}|_{C^0(A)}\sum_{A\in\mathcal{M}_0^j}|e^{S_jg}|_{C^0(A)}.$$

*Proof.* Fix $n,j\in\mathbb{N}$ with $j<n$. First, notice that

$$\sum_{A\in\mathcal{M}_0^n}|e^{S_ng}|_{C^0(A)}\geqslant\sum_{A\in\mathcal{M}_0^n}\sup_A e^{(S_{n-j}g+S_j^{-1}g)\circ T^j}\geqslant\sum_{A\in\mathcal{M}_{-j}^{n-j}}\sup_A e^{S_{n-j}g}\inf_A e^{S_j^{-1}g}$$

$$\geqslant\sum_{A\in\mathcal{M}_0^{n-j}}\sup_A e^{S_{n-j}g}\sum_{\substack{B\in\mathcal{M}_{-j}^0\\ B\cap A\neq\emptyset}}\inf_B e^{S_j^{-1}g}$$

$$\geqslant C_g\sum_{A\in\mathcal{M}_0^{n-j}}|e^{S_{n-j}g}|_{C^0(A)}\sum_{\substack{B\in\mathcal{M}_{-j}^0\\ B\cap A\neq\emptyset}}|e^{S_j^{-1}g}|_{C^0(B)}$$

$$\geqslant C_g\sum_{A\in\mathcal{M}_0^{n-j}}|e^{S_{n-j}g}|_{C^0(A)}\sum_{\substack{B\in\mathcal{M}_{-j}^0\\ B\cap A\neq\emptyset}}|e^{S_j^{-1}g}|_{C^0(B)},$$

where we used Lemma 3.2.3 for the forth inequality.

Recall that $L_u^{\delta_1}(\mathcal{M}_{-j}^0)$ denotes the elements of $\mathcal{M}_{-j}^0$ whose unstable diameter is longer than $\delta_1/3$. Similarly, $L_s^{\delta_1}(\mathcal{M}_0^{n-j})$ denotes those elements of $\mathcal{M}_0^{n-j}$ whose stable diameter is larger than $\delta_1/3$. By Lemma 3.3.4

$$\sum_{A\in L_s^{\delta_1}(\mathcal{M}_0^{n-j})}|e^{S_{n-j}g}|_{C^0(A)}\geqslant C_{n_1}\delta_1\sum_{A\in\mathcal{M}_0^{n-j}}|e^{S_{n-j}g}|_{C^0(A)},\quad\text{for } n-j\geqslant n_2.$$

Let $A \in L_s^{\delta_1}(\mathcal{M}_0^{n-j})$ and let $V_A \in \widehat{\mathcal{W}}^s$ be a stable curve in $A$ with length at least $\delta_1/3$. By Proposition 3.3.7,

$$\sum_{W_i \in \mathcal{G}_j^{\delta_0}(V_A)} |e^{S_j g}|_{C^0(W_i)} \geqslant c_0 \sum_{B \in \mathcal{M}_{-j}^0} |e^{S_j^{-1} g}|_{C^0(B)}.$$

Each component of $\mathcal{G}_j^{\delta_0}(V_A)$ corresponds to one component of $V_A \setminus \mathcal{S}_{-j}$ (up to subdivision of long pieces in $\mathcal{G}_j^{\delta_0}(V_A)$). Thus

$$\begin{aligned}\sum_{A \in \mathcal{M}_0^{n-j}} |e^{S_{n-j} g}|_{C^0(A)} \sum_{\substack{B \in \mathcal{M}_{-j}^0 \\ B \cap A \neq \emptyset}} |e^{S_j^{-1} g}|_{C^0(B)} &\geqslant \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^{n-j})} |e^{S_{n-j} g}|_{C^0(A)} \sum_{W_i \in \mathcal{G}_j^{\delta_0}(V_A)} |e^{S_j^{-1} g}|_{C^0(T^j W_i)} \\ &\geqslant \sum_{A \in L_s^{\delta_1}(\mathcal{M}_0^{n-j})} |e^{S_{n-j} g}|_{C^0(A)} \sum_{W_i \in \mathcal{G}_j^{\delta_0}(V_A)} |e^{S_j g}|_{C^0(W_i)} \\ &\geqslant C \sum_{A \in \mathcal{M}_0^{n-j}} |e^{S_{n-j} g}|_{C^0(A)} \sum_{B \in \mathcal{M}_{-j}^0} |e^{S_j^{-1} g}|_{C^0(B)},\end{aligned}$$

proving the lemma with $c_1 = c_0 C_{n_1} C^2 \delta_1$ when $n - j \geqslant n_2$. For $n - j \leqslant n_2$, since

$$\sum_{A \in \mathcal{M}_0^{n-j}} |e^{S_{n-j-1} g}|_{C^0(A)} \leqslant \left( \sum_{A \in \mathcal{M}_0^1} |e^{g}|_{C^0(A)} \right)^{n-j}$$

we obtain the lemma by decreasing $c_1$ since there are only finitely many values to correct for. □

**Proposition 3.3.10** (Exact Exponential Growth)**.** *Let $g$ be a $(\mathcal{M}_0^1, \alpha)$-Hölder continuous potential such that $P_*(T, g) - \sup g > 0$ and which has SSP.1. Let $c_1$ be the constant given by Lemma 3.3.9. Then for all $n \in \mathbb{N}$, we have*

$$\sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)} \leqslant \frac{2}{c_1} e^{n P_*(T, g)}.$$

*Proof.* Let $\psi(n) := e^{-nP_*(T,g)} \sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)}$. Suppose there exists $n_1 \in \mathbb{N}$ such that $\psi(n_1) \geqslant 2/c_1$, where $c_1$ is the constant from Lemma 3.3.9. Then

$$\psi(2n_1) \geqslant c_1 \psi(n_1)^2 = \frac{1}{c_1} (c_1 \psi(n_1))^2.$$

Integrating this bound, we have inductively for any $k \geqslant 1$,

$$\psi(2^k n_1) \geqslant \frac{1}{c_1} (c_1 \psi(n_1))^{2^k}.$$

This implies that $\lim_{k \to +\infty} \frac{1}{2^k n_1} \log \psi(2^k n_1) \geqslant \frac{1}{n_1} \log 2 > 0$, which contradicts the definition of $\psi(n)$ (since $\lim_{n \to +\infty} \frac{1}{n} \log \psi(n) = 0$). We conclude that $\psi(n) \leqslant 2/c_1$ for all $n \geqslant 1$. □

*Remark* 3.3.11. Notice that for $g = 0$, the condition $P_*(T, g) - \sup g > s_0 \log 2$ becomes $h_* > s_0 \log 2$, where $h_*$ is the topological entropy of $T$ defined in [BD20]. This is precisely the condition of sparse recurrence to singularities from [BD20], and as discussed there, we don't know any example of billiard table not satisfying this condition. Notice that by continuity, if $h_* > s_0 \log 2$ holds, then $P_*(T, g) - \sup g > s_0 \log 2$ holds for all $g$ in a neighbourhood of the zero potential. For potential $g$ close enough to 0, we have $\log \Lambda > \sup g - \inf g$. Therefore, by Lemmas 3.3.3, 3.3.4 and Corollary 3.3.6, there exists a neighbourhood of $g = 0$ (in the $(\mathcal{M}_0^1, \alpha)$-Hölder topology) in which every potential has SSP.1 and SSP.2, and thus all the consequent results from the present section also hold.

In particular, for any $t \in \mathbb{R}$ with $|t|$ close enough to zero, the potential $-t\tau$ has SSP.1 and SSP.2.

### 3.3.4 Estimates on norms of the potential

In Section 3.6, we will need similar estimates as in the present section but with the $C^0$ norm replaced by the $C^\beta$ norm, $0 < \beta < 1/3$. The following lemma shows that previous estimates are still valid up to a multiplicative constant.

**Lemma 3.3.12.** *For every bounded $(\mathcal{M}_0^1, \alpha)$-Hölder continuous potential $g$, there exists $C > 0$ such that for all $W \in \mathcal{W}^s$, all $n \geqslant 0$ and all $W_i \in \mathcal{G}_n^\delta(W)$, $|e^{S_n g}|_{C^\alpha(W_i)} \leqslant C|e^{S_n g}|_{C^0(W_i)}$, where $\delta \in (0, \delta_0]$.*

*Proof.* Let $g$ be such a potential. Let $c$ be such that $g \geqslant c$. Let $W_i \in \mathcal{G}_n^\delta(W)$. Then

$$\begin{aligned} H^\alpha_{W_i}(e^{S_n g}) &\leqslant \sum_{k=0}^{n-1} |e^{-g\circ T^k + S_n g}|_{C^0(W_i)} H^\alpha_{W_i}(g \circ T^k) \\ &\leqslant |e^{S_n g}|_{C^0(W_i)} \sum_{k=0}^{n-1} e^{-c} C \Lambda^{-\alpha k} |g|_{C^\alpha(M)} \\ &\leqslant |e^{S_n g}|_{C^0(W_i)} C \frac{1}{1 - \Lambda^\alpha} e^{-c} |g|_{C^\alpha(M)}, \end{aligned}$$

where for the second inequality we adapted the argument from [BD20, eq (6.2)], so that

$$\frac{g(T^k x) - g(T^k y)}{d_W(T^k x, T^k y)^\alpha} \frac{d_W(T^k x, T^k y)^\alpha}{d_W(x,y)^\alpha} \leqslant C H^\alpha_{T^k W_i}(g) |J^s T^k|^\alpha_{C^0(W_i)} \leqslant C \Lambda^{-\alpha k} |g|_{C^\alpha(M)}.$$

□

## 3.4 The Banach Spaces $\mathcal{B}$ and $\mathcal{B}_w$ and the Transfer Operators $\mathcal{L}_g$

In Section 3.6, we construct the equilibrium state $\mu_g$ for $T$ under the potential $g$ out of left and right eigenvectors, $\tilde{\nu}$ and $\nu$, of a transfer operator $\mathcal{L}_g$ associated with the billiard map and the potential $g$, acting on suitable Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ of anisotropic distributions. In this section, we define the Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ as well as the transfer operator $\mathcal{L}_g$.

### 3.4.1 Motivation and heuristics

The spaces $\mathcal{B}$ and $\mathcal{B}_w$ are the same as in [BD20], but we recall their construction not only for completeness, but also to introduce notations. The norms we introduce below are defined by integrating along stable manifolds in $\mathcal{W}^s$. We define precisely the notion of distance $d_{\mathcal{W}^s}(\cdot,\cdot)$ between such curves as well as a distance $d(\cdot,\cdot)$ defined among functions supported on these curves.

In the setup of uniform hyperbolic dynamics, the relevant transfer operator to study equilibrium states associated to a potential $g$ – see for example [Bal18] – can be defined on measurable function $f$ by

$$\mathcal{L}_g f = \left(e^g \frac{f}{J^s T}\right) \circ T^{-1}$$

where $J^sT$ is the stable Jacobian of $T$. Ignoring first the low regularity of $J^sT$, we see from the hyperbolicity of $T$ that the composition with $T^{-1}$ should increase the regularity of $f$ in the unstable direction, while decreasing the regularity in the stable direction. By integrating along stable manifolds against the arclength measure, we hope to recover some regularity along the stable manifold – notice that by a change of variable, $J^sT$ does disappear. Morally, the weak norm $|\cdot|_w$ and the strong stable norm $||\cdot||_s$ measure the regularity of the averaged action of $\mathcal{L}_g$. On the other hand, the strong unstable norm $||\cdot||_u$ captures the regularity when passing from a stable manifold to another one. Here, this regularity should be thought of as a log-scaled Hölder regularity.

### 3.4.2 Definition of the Banach spaces and embeddings into distribution

Recall that $\mathcal{W}^s$ denotes the set of all nontrivial connected subsets $W$ of stable manifolds for $T$ so that $W$ has length at most $\delta_0$. Such curves have curvature bounded above by a fixed constant [CM06, Prop. 4.29]. Thus $T^{-1}\mathcal{W}^s = \mathcal{W}^s$, up to subdivision of curves. Obviously, $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$. We define $\mathcal{W}^u$ similarly from unstable manifolds of $T$.

Given a curve $W \in \mathcal{W}^s$, we denote by $m_W$ the unnormalized Lebesgue (arclength) measure on $W$, so that $|W| = m_W(W)$. Since the stable cone $C^s$ (3.2.1) is bounded away from the vertical, we may view each stable manifolds $W \in \mathcal{W}^s$ as the graph of a function $\varphi_W(r)$ of the arclength coordinate $r$ ranging over some interval $I_W$, that is

$$W = \{G_W(r) \coloneqq (r, \varphi_W(r)) \mid r \in I_W\}.$$

Given two curves $W_1$, $W_2 \in \mathcal{W}^s$, we may use this representation to define a "distance" [7] between them. Define

$$d_{\mathcal{W}^s}(W_1, W_2) = |I_{W_1} \triangle I_{W_2}| + |\varphi_{W_1} - \varphi_{W_2}|_{C^1(I_{W_1} \cap I_{W_2})}$$

if $I_{W_1} \cap I_{W_2} \neq \emptyset$, and $d_{\mathcal{W}^s}(W_1, W_2) = +\infty$ otherwise.

Similarly, given two test functions $\psi_1$ on $W_1$, and $\psi_2$ on $W_2$, we define a distance between them by

$$d(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{C^0(I_{W_1} \cap I_{W_2})},$$

whenever $d_{\mathcal{W}^s}(W_1, W_2)$ is finite, and $d(\psi_1, \psi_2) = +\infty$ otherwise.

We can now introduce the norms used to define the spaces $\mathcal{B}$ and $\mathcal{B}_w$. These norms will depend on the constants $\epsilon_0 > 0$ and $\delta_0 \in (0, 1)$, as well as on four positive real numbers $\alpha$, $\beta$, $\gamma$ and $\zeta$ so that

$$0 < \beta < \alpha \leqslant \min\{1/3, \alpha_g\}, \quad 1 < 2^{s_0\gamma} < e^{P_*(T,g) - \sup g}, \quad 0 < \zeta < \gamma$$

where $g$ is a given, bounded $(\mathcal{M}_0^1, \alpha_g)$-Hölder potential such that $P_*(T, g) - \sup g > s_0 \log 2$.

7. Actually, $d_{\mathcal{W}^s}$ is not a metric since it does not satisfy the triangle inequality. It is nonetheless sufficient for our purpose to produce a usable notion of a distance between stable manifolds.

*Remark* 3.4.1. The condition $\alpha \leqslant 1/3$ is needed for [BD20, Lemma 4.4], which is used to prove the embedding into distributions. The number $1/3$ comes from the regularity of the density function of the conditional measures in the disintegration of $\mu_{\mathrm{SRB}}$ against the stable foliation. The bound $\alpha \leqslant \alpha_g$ will make possible to see $g$ as an element of $\mathcal{B}$. The upper bound on $\gamma$ arises from the use of the growth lemma 3.3.1. The dependence on $\delta_0$ comes from the definition of $\mathcal{W}^s$.

For $f \in C^1(M)$, define the weak norm of $f$ by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \leqslant 1}} \int_W f\, \psi \,\mathrm{d}m_W \,.$$

Similarly, define the strong stable norm of $f$ by [8]

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\beta(W) \\ |\psi|_{C^\beta(W)} \leqslant |\log|W||^\gamma}} \int_W f\, \psi \,\mathrm{d}m_W \,,$$

(note that $|f|_w \leqslant \max\{1, |\log \delta_0|^{-\gamma}\}\|f\|_s$). Finally, for $\varsigma \in (0, \gamma)$, define the strong unstable norm [9] of $f$ by

$$\|f\|_u = \sup_{\varepsilon \leq \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \leqslant \varepsilon}} \sup_{\substack{\psi_i \in C^\alpha(W_i) \\ |\psi_i|_{C^\alpha(W_i)} \leq 1 \\ d(\psi_1, \psi_2) = 0}} |\log \varepsilon|^\varsigma \left| \int_{W_1} f\, \psi_1 \,\mathrm{d}m_{W_1} - \int_{W_2} f\, \psi_2 \,\mathrm{d}m_{W_2} \right| .$$

In order to use functional analysis results, we need to work with complete spaces. Since $C^1(M)$ is not complete for the norms [10] $|\cdot|_w$ and $\|\cdot\|_s + \|\cdot\|_u$, we will use the corresponding completed spaces.

**Definition 3.4.2** (The Banach spaces)**.** *The space $\mathcal{B}_w$ is the completion of $C^1(M)$ with respect to the weak norm $|\cdot|_w$, while $\mathcal{B}$ is the completion of $C^1(M)$ with respect to the strong norm, $\|\cdot\|_{\mathcal{B}} = \|\cdot\|_s + \|\cdot\|_u$. Notice that since $|\cdot|_w \leqslant \|\cdot\|_{\mathcal{B}}$, there is a canonical map $\mathcal{B} \to \mathcal{B}_w$.*

Since the main purpose of the spaces $\mathcal{B}$ and $\mathcal{B}_w$ is to contain left and right eigenvectors of a transfer operator acting on those spaces, a crucial feature of $\mathcal{B}$ and $\mathcal{B}_w$ is that we can see them as subspaces of the distributional space $(C^1(M))^*$. Thanks to this property, we will be able to construct a positive distribution by pairing the left and right eigenvectors, and to extend it into the desired equilibrium measure. In order to state this result, we need to introduce some other spaces, on which the transfer operator will be naturally defined (and then extended to $\mathcal{B}$ and $\mathcal{B}_w$).

Define the usual homogeneity strips

$$\mathbb{H}_k := \left\{ (r, \varphi) \in M_i \mid \frac{\pi}{2} - \frac{1}{k^2} \leqslant \varphi \leqslant \frac{\pi}{2} - \frac{1}{(k+1)^2} \right\}, \quad k \geqslant k_0 \,,$$

8. The logarithmic modulus of continuity in $\|f\|_s$ is used to obtain a finite spectral radius.

9. The logarithmic modulus of continuity appears in $\|f\|_u$ because of the logarithmic modulus of continuity in $\|f\|_s$. Its presence in $\|f\|_u$ causes the loss of the spectral gap.

10. For example, the sequence $\left(\frac{1}{n} \sin 2\pi n^2 \frac{r}{|\Gamma_i|}\right)_n$ is a Cauchy sequence of $C^1(M)$ functions with respect to $|\cdot|_w$, but diverges in the $C^1$-norm.

and analogously for $k \leqslant -k_0$ (for $\varphi$ near $-\pi/2$). Define $\mathcal{W}^s_{\mathbb{H}} \subset \mathcal{W}^s$ as the set of stable manifolds $W \in \mathcal{W}^s$ such that $T^n W$ lies in a single homogeneity strip for all $n \geqslant 0$. We write $\psi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$ if $\psi \in C^\alpha(W)$ for all $W \in \mathcal{W}^s_{\mathbb{H}}$ with uniformly bounded Hölder norm. The norm of $\psi$ in $C^\alpha(\mathcal{W}^s_{\mathbb{H}})$ is defined to be the sup over all the $C^\alpha(W)$ norms, with $W$ ranging in $\mathcal{W}^s_{\mathbb{H}}$. Similarly, define the space $C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}})$ containing the functions $\psi$ such that $\psi \cos\varphi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$. The norm of $\psi$ in $C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}})$ is defined to be the norm of $\psi\cos\varphi$ in $C^\alpha(\mathcal{W}^s_{\mathbb{H}})$. Clearly, $C^\alpha(\mathcal{W}^s_{\mathbb{H}}) \subset C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}})$.

The canonical map $\mathcal{B}_w \to (\mathcal{F})^*$ (for $\mathcal{F} = C^1(M)$, or $\mathcal{F} = C^\alpha(\mathcal{W}^s_{\mathbb{H}})$) is understood in the following sense: for $f \in \mathcal{B}_w$, there exists $C_f < \infty$ such that letting $f_n \in C^1(M)$ be a sequence converging to $f$ in the $\mathcal{B}_w$ norm, for every $\psi \in \mathcal{F}$ the following limit exists

$$f(\psi) := \lim_{n\to+\infty} \int f_n \psi \, \mathrm{d}\mu_{\mathrm{SRB}}$$

and satisfies $|f(\psi)| \leqslant C_f ||\psi||_{\mathcal{F}}$.

We summarize the properties of these Banach spaces obtained in [BD20] in the following proposition.

**Proposition 3.4.3.** *The spaces $\mathcal{B}_w$ and $\mathcal{B}$ are such that:*
*(i) The following canonical maps are all continuous*

$$C^1(M) \to \mathcal{B} \to \mathcal{B}_w \to (C^\alpha(\mathcal{W}^s_{\mathbb{H}}))^* \to (C^1(M))^*,$$

*and the first two maps are injective. In particular, we also have the two injective and continuous maps*

$$(\mathcal{B}_w)^* \to \mathcal{B}^* \to (C^1(M))^*.$$

*(ii) The inclusion map $\mathcal{B} \hookrightarrow \mathcal{B}_w$ is compact.*

*Proof.* The point (i) is the content of [BD20, Proposition 4.2]. We detail the proof of the injectivity of the map $\mathcal{B} \to \mathcal{B}_w$. To do so, we prove that the formula defining $|\cdot|_w$ (respectively $||\cdot||_s$ and $||\cdot||_u$) can be extended when $f \in \mathcal{B}_w$ (respectively $f \in \mathcal{B}$), and that it coincides with the norm of $f$.

First, notice that when $f \in C^1(M)$, then for given $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ we have $\int_W f\psi \, \mathrm{d}m_W \leqslant |f|_w |\psi|_{C^\alpha(W)}$. Thus the map $f \mapsto \int_W f\psi \, \mathrm{d}m_W$ can be extended uniquely to $\mathcal{B}_w$.

Now, let $f \in \mathcal{B}_w$ and $\varepsilon > 0$. Let $f_n$ be a Cauchy sequence of $C^1(M)$ functions converging to $f$ in $\mathcal{B}_w$. Thus, there exists some $n_\varepsilon$ such that for all $n \geqslant n_\varepsilon$, $|f - f_n|_w \leqslant \varepsilon$. Let $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leqslant 1$. By definition of $|f_n|_w$, for all $n$, there exist $W_n$ and $\psi_n \in C^\alpha(W_n)$ with $|\psi_n|_{C^\alpha(W_n)} \leqslant 1$ such that

$$\left| \int_{W_n} f_n \psi_n \, \mathrm{d}m_{W_n} - |f_n|_w \right| \leqslant \varepsilon.$$

Thus, we have

$$\left| \int_{W_n} f\psi_n \, \mathrm{d}m_{W_n} - \int_{W_n} f_n \psi_n \, \mathrm{d}m_{W_n} \right| \leqslant |f - f_n|_w |\psi_n|_{C^\alpha(W_n)} \leqslant \varepsilon, \quad \forall n \geqslant n_\varepsilon,$$

and so $\left| |f_n|_w - \int_{W_n} f\psi_n \, \mathrm{d}m_{W_n} \right| \leqslant 2\varepsilon$. In particular, we get

$$\sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \leqslant 1}} \int_W f \, \psi \, \mathrm{d}m_W \geqslant |f|_w.$$

We now prove the reverse inequality. Using the same notations as above, there exist $V \in \mathcal{W}^s$ and $\varphi \in C^\alpha(V)$ with $|\varphi|_{C^\alpha(V)} \leqslant 1$ such that

$$\left| \int_V f\varphi \, \mathrm{d}m_V - \sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in C^\alpha(W)\\ |\psi|_{C^\alpha(W)}\leqslant 1}} \int_W f\,\psi\,\mathrm{d}m_W \right| \leqslant \varepsilon.$$

Now, since

$$\left| \int_V f_n\varphi\,\mathrm{d}m_V - \int_V f\varphi\,\mathrm{d}m_V \right| \leqslant |f-f_n|_w \leqslant \varepsilon, \quad \forall n \geqslant n_\varepsilon,$$

we have that $|\sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in C^\alpha(W)\\ |\psi|_{C^\alpha(W)}\leqslant 1}} \int_W f\,\psi\,\mathrm{d}m_W - \int_V f_n\varphi\,\mathrm{d}m_V| \leqslant 2\varepsilon$ for all large enough $n$. In particular

$$\sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in C^\alpha(W)\\ |\psi|_{C^\alpha(W)}\leqslant 1}} \int_W f\,\psi\,\mathrm{d}m_W \leqslant |f_n|_w + 2\varepsilon.$$

Taking the limit in $n$, we get the claimed inequality.

The corresponding results for $f \in \mathcal{B}$ and norms $||\cdot||_s$ and $||\cdot||_u$ are obtained similarly, noticing that for all $f \in C^1(M)$,

$$\int_W f\psi\,\mathrm{d}m_W \leqslant ||f||_s|\psi|_{C^\beta(W)}|\log|W||^{-\gamma} \leqslant ||f||_{\mathcal{B}}|\psi|_{C^\beta(W)}|\log|W||^{-\gamma}\,, \quad \forall W \in \mathcal{W}^s, \forall \psi \in C^\beta(W)$$

Thus the integrals against $C^\beta(W)$ functions in the definition of $||\cdot||_s$ makes sense even when $f \in \mathcal{B}$. On the other hand, since $|\cdot|_w \leqslant ||\cdot||_{\mathcal{B}}$, the integrals in the definition of $||\cdot||_u$ can be extended to $f \in \mathcal{B}$ as in the above case where $f \in \mathcal{B}_w$.

We can now show the injectivity of the canonical map $\mathcal{B} \to \mathcal{B}_w$. Let $f \in \mathcal{B}$ with $||f||_{\mathcal{B}} \neq 0$. If $||f||_s \neq 0$, then the fact that $|f|_w \neq 0$ follows from the definition of $C^\beta(W)$ as the closure of $C^1(W)$ in the $C^\beta$ norm, so that $C^\alpha(W)$ is dense in $C^\beta(W)$. Now, if $||f||_u \neq 0$, then by definition of $||\cdot||_u$, we can find some $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ so that $\int_w f\psi\,\mathrm{d}m_W > 0$. Thus $|f|_w \neq 0$.

The point (ii) is precisely the content of [BD20, Proposition 6.1]. □

### 3.4.3 The transfer operators

We may define the transfer operator $\mathcal{L}_g : (C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}}))^* \to (C^\alpha(\mathcal{W}^s))^*$, for a given weight function $g$ by

$$\mathcal{L}_g f(\psi) = f\big(e^g \tfrac{\psi\circ T}{J^sT}\big), \quad \psi \in C^\alpha(\mathcal{W}^s)\,.$$

This operator is well defined because, if $\psi \in C^\alpha(\mathcal{W}^s)$ then $e^g\,\psi\circ T \in C^\alpha(\mathcal{W}^s)$. Furthermore, since $J^sT$ and $\cos\varphi$ are 1/3-log-Hölder on homogeneous stable manifolds, and $\cos\varphi/J^sT$ is bounded away from 0 and $+\infty$ also on homogeneous stable manifolds, we get that $1/J^sT \in C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}})$. Thus $e^g\frac{\psi\circ T}{J^sT} \in C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}})$.

When $f \in C^1(M)$, we identify $f$ with the measure [11]

$$f\mu_{\mathrm{SRB}} \in (C^\alpha_{\cos}(\mathcal{W}^s_{\mathbb{H}}))^*\,. \tag{3.4.1}$$

11. To show the claimed inclusion just use that $\mathrm{d}\mu_{\mathrm{SRB}} = (2|\partial Q|)^{-1}\cos\varphi\,\mathrm{d}r\mathrm{d}\varphi$.

The measure above is (abusively) still denoted by $f$. For $f \in C^1(M)$, we have

$$\begin{aligned}\mathcal{L}_g(f\mu_{\text{SRB}})(\psi) &= \int f\, e^g \frac{\psi \circ T}{J^sT}\, \mathrm{d}\mu_{\text{SRB}} = \int \Big(e^g \frac{f}{J^sT}\Big) \circ T^{-1} \psi\, \mathrm{d}\mu_{\text{SRB}} \\ &= \Big(\Big(e^g \frac{f}{J^sT}\Big) \circ T^{-1}\, \mu_{\text{SRB}}\Big)(\psi).\end{aligned}$$

Thus, due to our identification (3.4.1) we have $\mathcal{L}_g f = (e^g f / J^sT) \circ T^{-1}$, as claimed above.

**Proposition 3.4.4.** *For any fixed $(\mathcal{M}_0^1, \alpha_g)$-Hölder potential $g$ and corresponding spaces $\mathcal{B}$ and $\mathcal{B}_w$:*

*(i) If $f \in C^1(M)$, then $\mathcal{L}_g(f\mu_{SRB}) \in \mathcal{B}$.*

*(ii) The operators $\mathcal{L}_g : (C^1(M), |\cdot|_w) \to \mathcal{B}_w$ and $\mathcal{L}_g : (C^1(M), ||\cdot||_{\mathcal{B}}) \to \mathcal{B}$ are continuous.*

*In particular, $\mathcal{L}_g$ extends uniquely into operators on both $\mathcal{B}_w$ and $\mathcal{B}$.*

Since the proof of Proposition 3.4.4(i) is particularly long, we dedicate the next subsection for its proof.

The proof of the second point (ii) follows from Proposition 3.5.1, in the case $n = 1$.

### 3.4.4 Proof of Proposition 3.4.4

The proof of point (i) of Proposition 3.4.4 is largely inspired from the proof of the analogous result [BD22, Lemma 4.3] (corresponding to the geometric potentials $g = -t \log J^uT$, $0 < t < t_*$ for some $t_* > 1$). In [BD22, Remark 4.11], Baladi and Demers explain how to adapt their proof to the case $g = 0$ (replacing the estimates using homogeneity layers by weaker ones, but not relying on homogeneity layers). Here, instead of adding a remark on top of another one, we give in complete details the proof in the case of a $\mathcal{M}_0^1$-Hölder potential $g$.

*Proof.* Let $f \in C^1(M)$. Since $M$ is compact, for a large enough constant $C$, $f + C > 0$. Now, by linearity, we have $\mathcal{L}_g(f) = \mathcal{L}_g(f + C) - \mathcal{L}_g(C)$. Thus, we only need to prove Proposition 3.4.4 in the two cases $f > 0$ and $f$ constant. The second case is a consequence of the first one when $f$ is a nonzero constant, whereas the case of $f = 0$ is immediate. Therefore, without loss of generality, we now assume that $f > 0$.

As in [BD22, Section 4.4], we introduce a mollification in order to approximate $\mathcal{L}_g(f)$ by $C^1(M)$ functions in the norm $||\cdot||_{\mathcal{B}}$: Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative, rotationally symmetric $C^\infty$ function, supported in the unit disk and such that $\int_{\mathbb{R}^2} \rho(z)\, \mathrm{d}z = 1$ and $|\rho|_{C^1} \leqslant 2$. For $\eta > 0$ define

$$f_\eta(x) := \int_{B_\eta(x)} \eta^{-2} \rho\Big(\frac{d(x,z)}{\eta}\Big) \mathcal{L}_g(f)(z)\, \mathrm{d}z,$$

where $B_\eta(x)$ is the ball of radius $\eta$ centred at $x$. Viewing the connected components of $M$ as subsets of $\mathbb{R}^2$ (up to some quotient), we set $\mathcal{L}_g(f) \equiv 0$ outside $M$ so that the integral is well defined even when $B_\eta(x) \not\subset M$. Since it will be convenient to have estimates depending on $\eta$ of the $C^0$ and $C^1$ norms of $f_\eta$, we start with those estimates.

*Estimations of $|f_\eta|_{C^0(M)}$ and $|f_\eta|_{C^1(M)}$.* Let $x \in M$ and $\eta > 0$ be such that $B_\eta(x) \cap T\mathcal{S}_0 \neq \emptyset$. Note that, because of the continuity of the flow, there can be at most $\tau_{\max}/\tau_{\min}+1$ connected components in $B_\eta(x) \setminus T\mathcal{S}_0$. Using the uniform transversality of $T\mathcal{S}_0$ with the stable cones, and the fact that there is a constant $C$ such that for all $k \geqslant k_0$, $k^2|J^sT| \in [C^{-1}, C]$ on $T\mathbb{H}_k$, we get that $T\mathbb{H}_k$ is at distance approximately $k^{-4}$ from $B_\eta(x) \cap T\mathcal{S}_0$. In particular, there is a constant $C'$ such that if $B_\eta(x) \cap T\mathbb{H}_k \neq \emptyset$, then $k \leqslant C'\eta^{-1/4}$. Similarly, from the estimate on the stable Jacobian, we get that $\operatorname{diam}^s(B_\eta(x) \cap T\mathbb{H}_k) \leqslant Ck^{-5}$. On the other hand, we easily get the bound $\operatorname{diam}^u(B_\eta(x) \cap T\mathbb{H}_k) \leqslant C\eta$. Putting together these estimates, we obtain

$$\begin{aligned} |f_\eta(x)| &\leqslant C \sum_{k \geqslant C'\eta^{-1/4}} \int_{B_\eta(x) \cap T\mathbb{H}_k} \eta^{-2} \mathcal{L}_g(f)(z) \, \mathrm{d}z \leqslant C \sum_{k \geqslant C'\eta^{-1/4}} (\eta^{-2} |e^g|_\infty |f|_\infty k^2)(C\eta \, Ck^{-5}) \\ &\leqslant C\eta^{-1} e^{\sup g} |f|_\infty \sum_{k \geqslant C'\eta^{-1/4}} k^{-3} \leqslant C\eta^{-1/2} e^{\sup g} |f|_\infty. \end{aligned}$$

We conclude that

$$|f_\eta|_{C^0(M)} \leqslant Ce^{\sup g}|f|_{C^0(M)}\eta^{-1/2}, \quad \text{and similarly,} \quad |f_\eta|_{C^1(M)} \leqslant Ce^{\sup g}|f|_{C^0(M)}\eta^{-3/2}. \tag{3.4.2}$$

*Estimation of* $||\mathcal{L}_g(f) - f_\eta||_s$. Fix $\eta > 0$ and let $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$ be such that $|\psi|_{C^\beta(W)} \leqslant |\log |W||^\gamma$. Since the stable and unstable cone fields are uniformly transverse, there is a constant $C_1 > 0$ such that for any $W' \in \mathcal{W}^s$ and $x' \in W'$, if $d^s(x', \partial W') \leqslant C_1\eta$ then $B_\eta(x') \subset M$. We distinguish two cases. Consider first the case $|W| \leqslant 2C_1\eta$. then

$$\begin{aligned} \int_W (\mathcal{L}_g(f) - f_\eta)\psi \, \mathrm{d}m_W &\leqslant Ce^{\sup g}|f|_\infty |\log|W||^\gamma (|T^{-1}W| + |W|\eta^{-1/2}) \\ &\leqslant Ce^{\sup g}|f|_\infty |\log|W||^\gamma (C|W|^{1/2} + |W|^{1/2}) \leqslant Ce^{\sup g}|f|_\infty |\log \eta|^\gamma \eta^{1/2}, \end{aligned} \tag{3.4.3}$$

where for the second inequality we used that $|T^{-1}W| \leqslant C|W|$, and for the third inequality that $t \mapsto t^{1/2}|\log t|^\gamma$ is increasing near $t = 0$.

Assume now that $|W| > 2C_1\eta$. Define $W_\eta^-$ to be $W$ minus the two subcurves of length $2C_1\eta$ of $W$ starting at each endpoint of $W$. Then, since $m_W(W \setminus W_\eta^-) \leqslant 4C_1\eta$, we can estimate as in the above case

$$\int_{W \setminus W_\eta^-} (\mathcal{L}_g(f) - f_\eta)\psi \, \mathrm{d}m_W \leqslant Ce^{\sup g}|f|_\infty \eta^{1/2} |\log \eta|^\gamma. \tag{3.4.4}$$

Since $W$ intersects at most $N = \frac{\tau_{\max}}{\tau_{\min}} + 1$ elements of $T\mathcal{S}_0$, the set $W \cap (\cup_{k \geqslant \eta^{-1/5}} T\mathbb{H}_k)$ is made of at most $N$ connected curves each of length at most $C\eta^{4/5}$. Estimating as above,

$$\int_{W \cap (\cup_{k \geqslant \eta^{-1/5}} T\mathbb{H}_k)} (\mathcal{L}_g(f) - f_\eta)\psi \, \mathrm{d}m_W \leqslant Ce^{\sup g}|f|_\infty \eta^{3/10} |\log \eta|^\gamma. \tag{3.4.5}$$

It remains to estimate to integral of $(\mathcal{L}_g(f) - f_\eta)\psi$ on the parts of $W_\eta^-$ that intersect $T\mathbb{H}_k$, for $k < \eta^{-1/5}$. To do so, we give an estimate of $\mathcal{L}_g(f) - f_\eta$ on those curves. Let $x \in W_\eta^- \cap T\mathbb{H}_{\bar{k}}$, for some $\bar{k} < \eta^{-1/5}$. Because of the uniform transversality between $T\mathcal{S}_0$ and the stable cones, for small enough $\eta$ fixed, not only $B_\eta(x)$ does not intersect $T\mathcal{S}_0$, but $B_\eta(x)$ lies in a bounded number of homogeneity strips. Note also that since $x \in W_\eta^-$, by definition of $C_1$, we have that $B_\eta(x) \subset M$).

Using [BD22, Lemma 4.9], one can find a constant $C > 0$ such that for all $\eta > 0$ and all $W \in \mathcal{W}^s$, there exists $W_u(\eta) \subset W$ such that for all $x' \in W_u(\eta)$, $x'$ cuts the unstable manifold passing through it into two curves of length at least $\eta$, and[12] $m_W(W \smallsetminus W_u(\eta)) \leqslant C\sqrt{\eta}$. Denote $S_\eta = W \smallsetminus W_u(\eta)$ and let $A_\eta(x) \subset B_\eta(x)$ be the subset of $B_\eta(x)$ foliated by unstable manifolds of length at least $2\eta$. Finally, denote $E_\eta(x) = B_\eta(x) \smallsetminus A_\eta(x)$. By construction, we have the following decomposition

$$\begin{aligned}
\mathcal{L}_g(f)(x) - f_\eta(x) &= \int_{B_\eta(x)} \eta^{-2}\rho\Big(\frac{d(x,z)}{\eta}\Big)(\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(z))\,\mathrm{d}z \\
&= \int_{A_\eta(x)} \eta^{-2}\rho\Big(\frac{d(x,z)}{\eta}\Big)(\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(z))\,\mathrm{d}z \\
&\quad + \int_{E_\eta(x)} \eta^{-2}\rho\Big(\frac{d(x,z)}{\eta}\Big)(\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(z))\,\mathrm{d}z\,.
\end{aligned} \tag{3.4.6}$$

We start by estimating the integral on $E_\eta(x)$. From the assumptions on $x$ and $\bar{k}$, $B_\eta(x)$ intersects only finitely many $T\mathbb{H}_k$ and $x \notin \cup_{k>\eta^{-1/5}} T\mathbb{H}_k$. Recall that, from the construction of homogeneity strips, we obtain the existence of a constant $C > 0$ such that for all $z \in B_\eta(x)$ we have

$$C^{-1} \leqslant \frac{J^sT(T^{-1}x)}{J^sT(T^{-1}z)} \leqslant C.$$

Furthermore, since $f$ is positive and bounded, there is a constant $C$ such that for all $z \in B_\eta(x)$ we have $|f(T^{-1}z)| \leqslant C|f(T^{-1}x)|$. Combining these last two estimates with the fact that $g$ is bounded, we get for all $z \in B_\eta(x)$,

$$\mathcal{L}_g(f)(z) = \mathcal{L}_g(f)(x)\frac{J^sT(T^{-1}x)}{J^sT(T^{-1}z)}\frac{e^{g(T^{-1}z)}}{e^{g(T^{-1}x)}}\frac{f(T^{-1}z)}{f(T^{-1}x)} \leqslant C\mathcal{L}_g(f)(x)\,. \tag{3.4.7}$$

As a consequence, we can estimate the integral over $E_\eta(x)$ as follows

$$\begin{aligned}
\int_{E_\eta(x)} \eta^{-2}\rho\Big(\frac{d(x,z)}{\eta}\Big)(\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(z))\,\mathrm{d}z &\leqslant C\eta^{-2}\mathcal{L}_g(f)(x)\int_{E_\eta(x)} \mathrm{d}z \\
&\leqslant C\eta^{-1}|S_\eta \cap B_\eta(x)|\mathcal{L}_g(f)(x)\,.
\end{aligned}$$

We now turn to the second term in the right hand side of (3.4.6). To do this, we split $A_\eta(x)$ into two subsets. For each point $y \in W_\eta^- \cap A_\eta(x)$ there exists an unstable manifold $U_y$ of length at least $2\eta$. Applying [BD22, Lemma 4.10] (with $\varrho = \eta^{5/4}$ in their notations), there exists a subset $U_y' \subset U_y$ such that $m_{U_y}(U_y \smallsetminus U_y') \leqslant C\eta^{5/4}$ and

$$\left|\frac{J^sT(z)}{J^sT(z')} - 1\right| \leqslant C_s\Big(\eta^{-5/6}d(z,z') + d(z,z')^{1/3}\Big)\,, \quad \forall z,z' \in U_y'\,,$$

where the constants $C$ and $C_s$ are independent of $U_y$ and $\eta$. Define $A_\eta'(x) \subset A_\eta(x)$ to be those points contained in sets $U_y'$. It follows from the properties of the sets $U_y'$ and the absolute continuity of the unstable foliation that

$$\int_{A_\eta(x)\smallsetminus A_\eta'(x)} \eta^{-2}\rho\Big(\frac{d(x,z)}{\eta}\Big)(\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(z))\,\mathrm{d}z \leqslant C\mathcal{L}_g(f)(x)\eta^{1/4} \tag{3.4.8}$$

where we again have used the bound $\mathcal{L}_g(f)(z) \leqslant \mathcal{L}_g(f)(x)$ on $B_\eta(x)$.

12. Actually, the $\sqrt{\eta}$ can be improved into $\eta^{4/5}$, but this weaker bound suffices.

On $A'_\eta(x)$, we estimate the difference $\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(z)$ as follows. Given $z \in A'_\eta(x)$, let $y \in W^-_\eta \cap A_\eta(x)$ be the point such that $z \in U'_y$, that is, the point of intersection of the stable manifold of $x$ with the unstable manifold of $z$. Therefore,

$$\begin{aligned}
&|\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(z)| \leqslant |\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(y)| + |\mathcal{L}_g(f)(y) - \mathcal{L}_g(f)(z)| \\
&\leqslant \left|\frac{e^{g(T^{-1}x)}}{J^sT(T^{-1}x)}\right| \left| f(T^{-1}x) - f(T^{-1}y) + \left(1 - \frac{e^{g(T^{-1}y)}}{e^{g(T^{-1}x)}} \frac{J^sT(T^{-1}x)}{J^sT(T^{-1}y)}\right) f(T^{-1}y)\right| \\
&\quad + \left|\frac{e^{g(T^{-1}y)}}{J^sT(T^{-1}y)}\right| \left| f(T^{-1}y) - f(T^{-1}z) + \left(1 - \frac{e^{g(T^{-1}z)}}{e^{g(T^{-1}y)}} \frac{J^sT(T^{-1}y)}{J^sT(T^{-1}z)}\right) f(T^{-1}z)\right| \\
&\leqslant \mathcal{L}_g(1)(x)\left( |f|_{C^1(M)} d(T^{-1}x, T^{-1}y) + |f|_\infty \left( \left|1 - \frac{J^sT(T^{-1}x)}{J^sT(T^{-1}y)}\right| + \left|1 - \frac{e^{g(T^{-1}y)}}{e^{g(T^{-1}x)}}\right| \right)\right) \\
&\quad + C\mathcal{L}_g(1)(x)\left( |f|_{C^1(M)} d(T^{-1}y, T^{-1}z) + |f|_\infty \left( \left|1 - \frac{J^sT(T^{-1}y)}{J^sT(T^{-1}z)}\right| + \left|1 - \frac{e^{g(T^{-1}z)}}{e^{g(T^{-1}y)}}\right| \right)\right) \\
&\leqslant C\mathcal{L}_g(1)(x)\Big[ |f|_{C^1(M)} d(T^{-1}x, T^{-1}y) + C|f|_\infty \Big( d(T^{-1}x, T^{-1}y)^{1/3} + |g|_{C^\alpha} d(T^{-1}x, T^{-1}y)^\alpha \Big) \\
&\quad + |f|_\infty \Big( C_s \Big( \eta^{-5/6} d(T^{-1}y, T^{-1}z) + d(T^{-1}y, T^{-1}z)^{1/3} \Big) + |g|_{C^\alpha} d(T^{-1}y, T^{-1}z)^\alpha \Big) \\
&\quad + |f|_{C^1(M)} d(T^{-1}y, T^{-1}z) \Big] .
\end{aligned}$$

Now, since $y$ and $z$ are on the same unstable manifold, we get that $d(T^{-1}y, T^{-1}z) \leqslant Cd(y,z) \leqslant C\eta$. On the other hand, because of the assumption on $x$ and $\bar{k}$, the subcurve of $W$ joining $x$ to $y$ is not cut by $T\mathcal{S}_0$, we can thus use the estimate on the stable Jacobian of $T$ to get

$$d(T^{-1}x, T^{-1}y) \leqslant C\bar{k}^2 d(x,y) \leqslant C\eta^{3/5},$$

since $\bar{k} < \eta^{-1/5}$. Putting these estimates together and keeping only the leading term in $\eta$, we obtain,

$$|\mathcal{L}_g(f)(x) - \mathcal{L}_g(f)(z)| \leqslant C|f|_{C^1(M)}|g|_{C^\alpha}\mathcal{L}_g(1)(x)\,\eta^{\alpha/6}, \quad \forall z \in A'_\eta(x). \tag{3.4.9}$$

Finally, integrating (3.4.9) on $A'_\eta(x)$, and recalling the decomposition (3.4.6), we get that

$$\begin{aligned}
|\mathcal{L}_g(f)(x) - f_\eta(x)| &\leqslant \mathcal{L}_g(1)(x)\Big( C|f|_\infty \eta^{1/4} + C|f|_\infty \eta^{-1}|S_\eta \cap B_\eta(x)| + C|f|_{C^1(M)}|g|_{C^\alpha}\eta^{\alpha/6}\Big) \\
&\leqslant C\mathcal{L}_g(1)(x)\Big( |f|_\infty \eta^{-1}|S_\eta \cap B_\eta(x)| + |f|_{C^1(M)}|g|_{C^\alpha}\eta^{\alpha/6}\Big).
\end{aligned} \tag{3.4.10}$$

We still need to integrate (3.4.10) against $\psi$ on $W^-_\eta \cap (\cup_{k \leqslant \eta^{-1/5}} T\mathbb{H}_k)$. We start with the second term on the right hand side of (3.4.10).

$$\begin{aligned}
\int_{W^-_\eta \cap (\cup_{k\leqslant\eta^{-1/5}} T\mathbb{H}_k)} C|f|_{C^1(M)}|g|_{C^\alpha}\eta^{\alpha/6}\mathcal{L}_g(1)\psi \, \mathrm{d}m_W &\leqslant C|f|_{C^1(M)}|g|_{C^\alpha}\eta^{\alpha/6}|\psi|_{C^0(W)}C\sqrt{|W|} \\
&\leqslant C|f|_{C^1(M)}|g|_{C^\alpha}\eta^{\alpha/6}|W|^{1/2}|\log|W||^\gamma \leqslant C|f|_{C^1(M)}|g|_{C^\alpha}\eta^{\alpha/6}.
\end{aligned} \tag{3.4.11}$$

Now for the first term in (3.4.10). Let $I_\eta(x) = B_\eta(x) \cap W$, and use the parametrization $W = G_W(x, I_{W,x}) = \{(r, \varphi_W(x,r)) \mid r \in I_{W,x}\}$ such that $G_W(x,0) = x$. Thus

$$\begin{aligned}
|S_\eta \cap B_\eta(x)| &= \int_{I_\eta(x)} \mathbb{1}_{S_\eta}(z) \, \mathrm{d}m_W(z) \\
&= \int_{-2\eta}^{2\eta} \mathbb{1}_{S_\eta}(G_W(x,r)) JG_W(x,r) \, \mathrm{d}r.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{W_\eta^- \cap (\cup_{k\leqslant \eta^{-1/5}} T\mathbb{H}_k)} C|f|_\infty \mathcal{L}_g(1)(x)\frac{\psi(x)}{\eta}|S_\eta \cap B_\eta(x)|\,\mathrm{d}m_W(x) \\
&\quad = C|f|_\infty \int_{W_\eta^- \cap (\cup_{k\leqslant \eta^{-1/5}} T\mathbb{H}_k)} \mathcal{L}_g(1)(x)\frac{\psi(x)}{\eta}\int_{-2\eta}^{2\eta} \mathbb{1}_{S_\eta}(G_W(x,r))JG_W(x,r)\,\mathrm{d}r\mathrm{d}m_W(x) \\
&\quad \leqslant C|f|_\infty \frac{|\log|W||^\gamma}{\eta}\int_{-2\eta}^{2\eta}\int_{W_\eta^-} \mathcal{L}_g(1)(x)\mathbb{1}_{S_\eta}(G_W(x,r))JG_W(x,r)\,\mathrm{d}m_W(x)\mathrm{d}r \\
&\quad \leqslant C|f|_\infty e^{\sup g}\frac{|\log|W||^\gamma}{\eta}\int_{-2\eta}^{2\eta}|S_\eta|^{1/2}\,\mathrm{d}r \leqslant 4C|f|_\infty e^{\sup g}\eta^{1/2}|\log\eta|^\gamma
\end{aligned}$$

where we used that translations of $W_\eta^-$ up to length $C_1\eta$ (here $2\eta$, but we can assume that $C_1 > 2$) are subsets of $W$, and then that $|T^{-1}S_\eta| \leqslant C|S_\eta|^{1/2}$. Putting together this estimate with (3.4.11), we obtain

$$\int_{W_\eta^- \cap (\cup_{k\leqslant \eta^{-1/5}} T\mathbb{H}_k)} (\mathcal{L}_g(f) - f_\eta)\psi\,\mathrm{d}m_W \leqslant C|f|_{C^1(M)}(|g|_{C^\alpha} + e^{\sup g})\eta^{\alpha/6}.$$

Combining this last estimate with (3.4.4) and (3.4.5) as well as with (3.4.3), we get that for any $W \in \mathcal{W}^s$, $|\psi|_{C^\beta(W)} < 1$ and any $\eta > 0$,

$$\int_W (\mathcal{L}_g(f) - f_\eta)\psi\,\mathrm{d}m_W \leqslant C|f|_{C^1}(|g|_{C^\alpha} + e^{\sup g})\eta^{\alpha/6}. \tag{3.4.12}$$

Taking the appropriate supremum over $\psi$, we get that $||\mathcal{L}_g(f) - f_\eta||_s \leqslant C|f|_{C^1}(|g|_{C^\alpha} + e^{\sup g})\eta^{\alpha/6}$, which converges to 0 as $\eta$ goes to zero.

*Estimation of* $||\mathcal{L}_g(f) - f_\eta||_u$. Let $0 < \varepsilon < \varepsilon_0$, and $W_1$, $W_2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W_1, W_2) \leqslant \varepsilon$. Let $\psi_i \in C^\alpha(W_i)$ with $|\psi_i|_{C^\alpha(W_i)} \leqslant 1$, and $d(\psi_1,\psi_2) = 0$. We want to estimate

$$|\log\varepsilon|^\zeta\left|\int_{W_1}(\mathcal{L}_g(f) - f_\eta)\psi_1\,\mathrm{d}m_{W_1} - \int_{W_2}(\mathcal{L}_g(f) - f_\eta)\psi_2\,\mathrm{d}m_{W_2}\right|.$$

We distinguish two cases. First, assume that $\eta^{\alpha/6} < |\log\varepsilon|^{-2\zeta}$. Then, applying twice (3.4.12), we obtain

$$\begin{aligned}
&|\log\varepsilon|^\zeta\left|\int_{W_1}(\mathcal{L}_g(f) - f_\eta)\psi_1\,\mathrm{d}m_{W_1} - \int_{W_2}(\mathcal{L}_g(f) - f_\eta)\psi_2\,\mathrm{d}m_{W_2}\right| \\
&\qquad \leqslant C\eta^{-\alpha/12}\eta^{\alpha/6}(|g|_{C^\alpha} + e^{\sup g})|f|_{C^1(M)} \leqslant C\eta^{\alpha/12}(|g|_{C^\alpha} + e^{\sup g})|f|_{C^1(M)}.
\end{aligned} \tag{3.4.13}$$

Assume now that $\eta^{\alpha/6} \geqslant |\log\varepsilon|^{-2\zeta}$. We decompose the difference of integrals as follows

$$\begin{aligned}
&\int_{W_1}(\mathcal{L}_g(f) - f_\eta)\psi_1\,\mathrm{d}m_{W_1} - \int_{W_2}(\mathcal{L}_g(f) - f_\eta)\psi_2\,\mathrm{d}m_{W_2} \\
&\quad = \left(\int_{W_1}\mathcal{L}_g(f)\psi_1\,\mathrm{d}m_{W_1} - \int_{W_2}\mathcal{L}_g(f)\psi_2\,\mathrm{d}m_{W_2}\right) + \left(\int_{W_2} f_\eta\psi_2\,\mathrm{d}m_{W_2} - \int_{W_1} f_\eta\psi_1\,\mathrm{d}m_{W_1}\right).
\end{aligned} \tag{3.4.14}$$

We estimate the two differences of (3.4.14) separately. We start with the difference involving $f_\eta$. Using the notations from the definition of $d_{\mathcal{W}^s}$, we see the curves $W_\ell$, $\ell = 1, 2$, as the graph of the functions $r \in I_\ell \mapsto \varphi_{W_\ell}(r)$. Since the at most two curves $V_i^\ell \subset W_\ell$

corresponding to intervals $I_1 \triangle I_2$ have length at most $C\varepsilon$ by definition of $d_{\mathcal{W}^s}(W_1, W_2) \leqslant \varepsilon$, we obtain

$$\int_{V_i^\ell} f_\eta \psi_\ell \, \mathrm{d}m_{W_\ell} \leqslant C\varepsilon |f_\eta|_{C^0(M)} \tag{3.4.15}$$

On the other hand, for the integrals over the curves $U_\ell \subset W_\ell$ corresponding to $I_1 \cap I_2$, we first notice that $|I_1 \cap I_2|$ is uniformly bounded. Thus, doing a change of variable with $G_{U_\ell}(r) = (r, \varphi_{U_\ell}(r))$, we obtain

$$\int_{U_1} f_\eta \psi_1 \, \mathrm{d}m_{W_1} - \int_{U_2} f_\eta \psi_2 \, \mathrm{d}m_{W_2} \leqslant C |(f_\eta \psi_1) \circ G_{U_1} \, JG_{U_1} - (f_\eta \psi_2) \circ G_{U_2} \, JG_{U_2}|_{C^0(I_1 \cap I_2)} \tag{3.4.16}$$

Now, using the definition of $d_{\mathcal{W}^s}(W_1, W_2) \leqslant \varepsilon$, we get that $|JG_{U_1} - JG_{U_2}|_{C^0(I_1 \cap I_2)} \leqslant \varepsilon$ as well as $d(G_{U_1}(r), G_{U_2}(r)) \leqslant \varepsilon$ for all $r \in I_1 \cap I_2$. Also, since $\psi_1 \circ G_{U_1} = \psi_2 \circ G_{U_2}$, we can write

$$\begin{aligned} &|(f_\eta \psi_1) \circ G_{U_1} \, JG_{U_1} - (f_\eta \psi_2) \circ G_{U_2} \, JG_{U_2}| \\ &\quad = |((f_\eta \psi_1) \circ G_{U_1})(JG_{U_1} - JG_{U_2}) + (\psi_2 \circ G_{U_2} \, JG_{U_2})(f_\eta \circ G_{U_1} - f_\eta \circ G_{U_2})| \\ &\quad \leqslant C\varepsilon |f_\eta|_{C^0(M)} + C\varepsilon |f_\eta|_{C^1(M)} \leqslant C\varepsilon |f_\eta|_{C^1(M)} \end{aligned}$$

Combining this estimate with (3.4.16), (3.4.15) and (3.4.2), we obtain

$$\begin{aligned} |\log \varepsilon|^\zeta \Big| \int_{W_2} f_\eta \psi_2 \, \mathrm{d}m_{W_2} - \int_{W_2} f_\eta \psi_2 \, \mathrm{d}m_{W_2} \Big| &\leqslant C\varepsilon |f_\eta|_{C^1(M)} |\log \varepsilon|^\zeta \\ &\leqslant C e^{\sup g} |f|_{C^0(M)} \eta^{-3/2} \eta^{-\alpha/12} \exp(-\eta^{-\frac{\alpha}{12\zeta}}). \end{aligned} \tag{3.4.17}$$

We now pass to the estimate of the difference involving $\mathcal{L}_g(f)$ in (3.4.14), still assuming that $\eta^{\alpha/6} \geqslant |\log \varepsilon|^{-2\zeta}$. We perform the same decomposition of $T^{-1} W_\ell$ into matched curves $U_j^\ell$ and unmatched curves $V_i^\ell$ as in the proof of Proposition 3.5.1, but for $n = 1$. Then, decomposing the difference of integrals

$$\begin{aligned} \Big| \int_{W_1} \mathcal{L}_g(f) \psi_1 \, \mathrm{d}m_{W_1} - \int_{W_2} \mathcal{L}_g(f) \psi_2 \, \mathrm{d}m_{W_2} \Big| &\leqslant \sum_{\ell, i} \Big| \int_{V_i^\ell} f \psi_\ell \circ T e^g \, \mathrm{d}m_{W_\ell} \Big| \\ &\quad + \sum_j \Big| \int_{U_j^1} f \psi_1 \circ T e^g \, \mathrm{d}m - \int_{U_j^2} f \psi_2 \circ T e^g \, \mathrm{d}m \Big|. \end{aligned}$$

We first estimate the part involving the unmatched pieces $V_i^\ell$. We only need to control the sum over $i$ of $|V_i^\ell|$. Since, by construction, we have $|TV_i^\ell| \leqslant C\varepsilon$, we thus have $|V_i^\ell| \leqslant C\varepsilon^{1/2}$. Now, since there is at most two curves $V_i^\ell$ in each element of $\mathcal{G}_1^{\delta_0}(W_\ell)$, we use Lemma 3.3.1(b) with $\gamma = 0$ to obtain that

$$\sum_{\ell, i} \Big| \int_{V_i^\ell} f \psi_\ell \circ T e^g \, \mathrm{d}m_{W_\ell} \Big| \leqslant |f|_\infty e^{\sup g} \sum_{\ell, i} |V_i^\ell| \leqslant C |f|_\infty e^{\sup g} \# \mathcal{M}_0^1 \, \varepsilon^{1/2}$$

In order to bound the sum involving the matched curves $U_j^\ell$, we decompose each difference of integrals introducing $\phi_j := (e^g \psi_1 \circ T) \circ G_{U_j^1} \circ G_{U_j^2}^{-1}$,

$$\begin{aligned} \int_{U_j^1} f \psi_1 \circ T e^g \, \mathrm{d}m - \int_{U_j^2} f \psi_2 \circ T e^g \, \mathrm{d}m &= \int_{U_j^1} f \psi_1 \circ T e^g \, \mathrm{d}m - \int_{U_j^2} f \phi_j \, \mathrm{d}m \\ &\quad + \int_{U_j^2} f (\phi_j - \psi_2 \circ T \, e^g) \, \mathrm{d}m. \end{aligned} \tag{3.4.18}$$

Using the computations from the proof of Proposition 3.5.1 in the case $n = 1$, the last term can be estimated as follows

$$\sum_j \Big| \int_{U_j^2} f(\phi_j - \psi_2 \circ T\, e^g)\, \mathrm{d}m \Big| \leqslant C\varepsilon^{\alpha-\beta} ||f||_s |\log \delta_0| \sum_{A \in \mathcal{M}_0^n} |e^g|_{C^0(A)}.$$

For difference in (3.4.18), we use a change of variable in order to integrate on $I_{U_j^1} = I_{U_j^2}$.

$$\int_{U_j^1} f\psi_1 \circ Te^g \,\mathrm{d}m - \int_{U_j^2} f\phi_j \,\mathrm{d}m = \int_{I_{U_j^1}} (\psi_1 \circ Te^g) \circ G_{U_j^1} (f \circ G_{U_j^1} JG_{U_j^1} - f \circ G_{U_j^2} JG_{U_j^2})\,\mathrm{d}r$$

We now give an upper bound on the function inside the integral. Obviously, $(\psi_1 \circ Te^g) \circ G_{U_j^1} \leqslant e^{\sup g}$. On the other hand, from [DZ11, Lemma 4.2], there exists $C > 0$ independent of $W_1$ and $W_2$ such that, for all $j$, $d_{\mathcal{W}^s}(U_j^1, U_j^2) \leqslant C\varepsilon$. In particular, $|JG_{U_j^1} - JG_{U_j^2}|_{C^0(I_{U_j^1})} \leqslant C\varepsilon$ and $d(G_{U_j^1}(r), G_{U_j^2}(r)) \leqslant C\varepsilon$ for all $r \in I_{U_j^1}$. Thus

$$|f \circ G_{U_j^1} JG_{U_j^1} - f \circ G_{U_j^2} JG_{U_j^2}|(r) \leqslant C|f|_{C^1(M)}\varepsilon,$$

where we used that $|JG_{U_j^\ell}| \leqslant C$ for some constant $C$ uniform on $\mathcal{W}^s$. Now, since there is at most one matched piece $U_j^\ell$ per element of $\mathcal{G}_1^{\delta_0}(W_\ell)$, we use Lemma 3.3.1(b) with $\gamma = 0$ in order to sum over $j$. Therefore

$$\sum_j \Big| \int_{U_j^1} f\psi_1 \circ Te^g \,\mathrm{d}m - \int_{U_j^2} f\phi_j \,\mathrm{d}m \Big| \leqslant Ce^{\sup g}|f|_{C^1(M)}\varepsilon \# \mathcal{M}_0^1.$$

Combining the estimates on matched and unmatched pieces, we finally obtain

$$\Big| \int_{W_1} \mathcal{L}_g(f)\psi_1 \,\mathrm{d}m_{W_1} - \int_{W_2} \mathcal{L}_g(f)\psi_2 \,\mathrm{d}m_{W_2} \Big| \leqslant C\varepsilon^{\min(\alpha-\beta, 1/2)} |f|_{C^1(M)} e^{\sup g} \# \mathcal{M}_0^1 \quad (3.4.19)$$

Combining (3.4.17) with (3.4.19) inside (3.4.14), we get

$$\begin{aligned} &|\log \varepsilon|^\varsigma \Big| \int_{W_1} (\mathcal{L}_g(f) - f_\eta)\psi_1 \,\mathrm{d}m_{W_1} - \int_{W_2} (\mathcal{L}_g(f) - f_\eta)\psi_2 \,\mathrm{d}m_{W_2} \Big| \\ &\qquad \leqslant Ce^{\sup g}|f|_{C^1(M)} \eta^{-3/2-\alpha/12} \exp(-\min(\alpha-\beta, 1/2)\eta^{-\alpha/12\varsigma}) \end{aligned} \quad (3.4.20)$$

Finally, from (3.4.13) and (3.4.20), and taking the appropriate supremums, on get an upperbound on $||f_\eta - \mathcal{L}_g(f)||_u$ that converges to 0 as $\eta$ goes to zero. Thus, we have shown that

$$||f_\eta - \mathcal{L}_g(f)||_{\mathcal{B}} = ||f_\eta - \mathcal{L}_g(f)||_s + ||f_\eta - \mathcal{L}_g(f)||_u \xrightarrow[\eta \to 0]{} 0.$$

□

## 3.5 Norm Estimates and Spectral Radius

The purpose of this section is to state and prove sharp upper and lower bounds on the norm of the iterated operator $\mathcal{L}_g^n$, both in $\mathcal{B}_w$ and $\mathcal{B}$.

**Proposition 3.5.1.** *Let $g$ be a $(\mathcal{M}_0^1, \alpha_g)$-Hölder continuous potential. Assume that $P_*(T,g) - \sup g > s_0 \log 2$ and that SSP.1 holds. Then there exist $\delta_0$ and $C > 0$ such that for all $f \in \mathcal{B}$,*

$$|\mathcal{L}_g^n f|_w \leqslant \frac{C}{\delta_0} e^{nP_*(T,g)} |f|_w \,, \quad \forall n \geq 0 \,; \tag{3.5.1}$$

$$\|\mathcal{L}_g^n f\|_s \leqslant \frac{C}{\delta_0} e^{nP_*(T,g)} \|f\|_s \,, \quad \forall n \geq 0 \,; \tag{3.5.2}$$

$$\|\mathcal{L}_g^n f\|_u \leqslant \frac{C}{\delta_0} (\|f\|_u + \|f\|_s) e^{nP_*(T,g)} \,, \quad \forall n \geq 0 \,. \tag{3.5.3}$$

*It follows that the spectral radius of $\mathcal{L}_g$ on $\mathcal{B}$ and $\mathcal{B}_w$ is at most $e^{P_*(T,g)}$.*

*Remark* 3.5.2. It is possible to obtain similar estimates without the assumption SSP.1, however an additional factor $e^{n\varepsilon}$ appears on the right hand sides, for any arbitrary $\varepsilon > 0$. We indicate places in the proof where it happens and how to correct for it. The conclusion about the upper bound of the spectral radius still holds. Nonetheless, in order to construct nontrivial maximal eigenvectors, we will need the estimates from Proposition 3.5.1.

**Theorem 3.5.3.** *Let $g$ be a $(\mathcal{M}_0^1, \alpha_g)$-Hölder continuous potential. Assume that $P_*(T,g) - \sup g > s_0 \log 2$ and that SSP.1 holds. Then there exists $C$ such that*

$$||\mathcal{L}_g^n 1||_s \geqslant |\mathcal{L}_g^n 1|_w \geqslant C \frac{\delta_1}{2} e^{nP_*(T,g)}.$$

*Proof of Proposition 3.5.1.* Let $\delta_0$ be the scale associated to $g$ as in the beginning of Section 3.3.2. The set $\mathcal{W}^s$ is defined with respect to the scale $\delta_0$.

We start with the weak norm estimate (3.5.1). Let $f \in C^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ be such that $|\psi|_{C^\alpha(W)} \leqslant 1$. For $n \geqslant 0$ we use the definition of the weak norm on each $W_i \in \mathcal{G}_n^{\delta_0}(W)$ to estimate

$$\int_W \mathcal{L}_g^n f \psi \, \mathrm{d}m_W = \sum_{W_i \in \mathcal{G}_n^{\delta_0}(W)} \int_{W_i} f e^{S_n g} \psi \circ T^n \mathrm{d}m_{W_i} \leqslant |f|_w \sum_{W_i \in \mathcal{G}_n^{\delta_0}(W)} |e^{S_n g}|_{C^\alpha(W_i)} |\psi \circ T^n|_{C^\alpha(W_i)}.$$

Clearly, $\sup |\psi \circ T^n|_{W_i} \leqslant \sup_W |\psi|$. For $x, y \in W_i$, we have,

$$\begin{aligned} \frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(T^n x, T^n y)^\alpha} \cdot \frac{d_W(T^n x, T^n y)^\alpha}{d_W(x,y)^\alpha} &\leqslant C |\psi|_{C^\alpha(W)} |J^s T^n|_{C^0(W_i)}^\alpha \\ &\leqslant C \Lambda^{-\alpha n} |\psi|_{C^\alpha(W)} \,, \end{aligned} \tag{3.5.4}$$

so that $H^\alpha_{W_i}(\psi \circ T^n) \leqslant C \Lambda^{-\alpha n} H^\alpha_W(\psi)$ and thus $|\psi \circ T^n|_{C^\alpha(W_i)} \leqslant C |\psi|_{C^\alpha(W)}$. By Lemma 3.3.12, we get

$$\begin{aligned} \int_W \mathcal{L}_g^n f \psi \, \mathrm{d}m_W &\leqslant C |f|_w |\psi|_{C^\alpha(W)} \sum_{W_i \in \mathcal{G}_n^{\delta_0}(W)} |e^{S_n g}|_{C^0(W_i)} \leqslant \frac{2C}{\delta_0} |f|_w |\psi|_{C^\alpha(W)} \sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)}, \\ &\leqslant \frac{2C}{c_1 \delta_0} |f|_w |\psi|_{C^\alpha(W)} e^{nP_*(T,g)}, \end{aligned}$$

where the second inequality uses that there are no more than $2\delta_0^{-1}$ curves $W_i$ of $\mathcal{G}_n^{\delta_0}(W)$ per element of $\mathcal{M}_0^n$, and the third inequality uses the Exact Exponential Growth from Proposition 3.3.10 [13].

13. without the assumptions SSP.1 and SSP.3, Proposition 3.3.10 might not hold. Still, for $\varepsilon > 0$ and all $n \geqslant 1$, $\sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)} \leqslant C_\varepsilon \, e^{n(P_*(T,g)+\varepsilon)}$ because of the subadditivity from Theorem 3.2.1.

Now we prove the strong stable norm estimate (3.5.2). We can choose $m$ so large that $2^{s_0\gamma}(Km+1)^{1/m} < e^{P_*(T,g)-\sup g}$. Let $W \in \mathcal{W}^s$, $\psi \in C^\beta(W)$ such that $|\psi|_{C^\beta(W)} \leqslant |\log|W||^\gamma$. Then, by definition of the strong norm

$$\begin{aligned}\int_W \mathcal{L}_g^n f\psi \mathrm{d}m_W &= \sum_{W_i\in\mathcal{G}_n^{\delta_0}(W)} \int_{W_i} f\psi\circ T^n e^{S_n g}\mathrm{d}m_{W_i} \\ &\leqslant \sum_{W_i\in\mathcal{G}_n^{\delta_0}(W)} ||f||_s|\psi\circ T^n|_{C^\beta(W_i)}|e^{S_n g}|_{C^\beta(W_i)}|\log|W_i||^{-\gamma} \\ &\leqslant C||f||_s \sum_{W_i\in\mathcal{G}_n^{\delta_0}(W)} \left(\frac{\log|W|}{\log|W_i|}\right)^\gamma |e^{S_n g}|_{C^\beta(W_i)} \\ &\leqslant C||f||_s 2^{2\gamma+1}\delta_0^{-1}\sum_{j=1}^n 2^{js_0\gamma}(Km+1)^{j/m}e^{j\sup g}\sum_{A\in\mathcal{M}_0^{n-j}}|e^{S_{n-j}g}|_{C^0(A)}\end{aligned}$$

where for the last line we used Lemma 3.3.1(b) and Lemma 3.3.12. Let

$$D_n := C2^{2\gamma+1}\delta_0^{-1}\sum_{j=1}^n 2^{js_0\gamma}(Km+1)^{j/m}e^{j\sup g}\sum_{A\in\mathcal{M}_0^{n-j}}|e^{S_{n-j}g}|_{C^0(A)}.$$

From Proposition 3.3.10, for all $n \geqslant 1$, $\sum_{A\in\mathcal{M}_0^n}|e^{S_n g}|_{C^0(A)} \leqslant \frac{2}{c_1}e^{nP_*(T,g)}$. Let $\varepsilon_1 = P_*(T,g) - \sup g - \log(2^{s_0\gamma}(Km+1)^{1/m})$. Thus [14],

$$D_n \leqslant 2^{2\gamma+1}\frac{C}{c_1\delta_0}\sum_{j=1}^n e^{(P_*(T,g)-\varepsilon_1)j}e^{(n-j)P_*(T,g)} \leqslant 2^{2\gamma+1}\frac{1}{1-e^{-\varepsilon_1}}\frac{C}{c_1\delta_0}e^{nP_*(T,g)}.$$

This concludes the proof of (3.5.2).

Finally, we now prove the strong unstable norm estimate (3.5.3). Fix $\tilde\varepsilon < \varepsilon_0$, and consider two curves $W^1$, $W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) < \tilde\varepsilon$.

For $n \geqslant 1$, we describe how to partition $T^{-n}W^\ell$ into "matched" pieces $U_j^\ell$ and "unmatched" pieces $V_i^\ell$, $\ell = 1, 2$.

Let $\omega$ be a connected component of $W^1 \setminus \mathcal{S}_{-n}$. To each point $x \in T^{-n}\omega$, we associate a vertical line segment $\gamma_x$ of length at most $C\Lambda^{-n}\tilde\varepsilon$ such that its image $T^n\gamma_x$, if not cut by a singularity, will have length $C\tilde\varepsilon$. By [CM06, §4.4], all the tangent vectors to $T^i\gamma_x$ lie in the unstable cone $C^u(T^ix)$ for each $i \geqslant 1$ so that they remain uniformly transverse to the stable cone and enjoy the minimum expansion given by $\Lambda$.

Doing this for each connected component of $W^1 \setminus \mathcal{S}_{-n}$, we subdivide $W^1 \setminus \mathcal{S}_{-n}$ into a countable collection of subintervals of points for which $T^n\gamma_x$ intersects $W^2 \setminus \mathcal{S}_{-n}$ and subintervals for which this is not the case. This in turn induces a corresponding partition on $W^2 \setminus \mathcal{S}_{-n}$.

We denote by $V_i^\ell$ the pieces in $T^{-n}W^\ell$ which are not matched up by this process and note that the images $T^nV_i^\ell$ occur either at the endpoints of $W^\ell$ or because the vertical segment $\gamma_x$ has been cut by a singularity. In both cases, the length of the curves $T^nV_i^\ell$ can be at most $C\tilde\varepsilon$ due to the uniform transversality of $\mathcal{S}_{-n}$ with the stable cone and of $C^s(x)$ with $C^u(x)$.

14. Here, again, the conclusion from Proposition 3.3.10 can be replaced.

In the remaining pieces the foliation $\{T^n\gamma_x\}_{x\in T^{-n}W^1}$ provides a one-to-one correspondence between points in $W^1$ and $W^2$. We further subdivide these pieces in such a way that the lengths of their images under $T^{-i}$ are less than $\delta_0$ for each $0 \leqslant i \leqslant n$ and the pieces are pairwise matched by the foliation $\{\gamma_x\}$. We call these matched pieces $U_j^\ell$. Since the stable cone is bounded away from the vertical direction, we can adjust the elements of $\mathcal{G}_n^{\delta_0}(W^\ell)$ created by artificial subdivisions due to length so that $U_j^\ell \subset W_i^\ell$ and $V_k^\ell \subset W_{i'}^\ell$ for some $W_i^\ell, W_{i'}^\ell \in \mathcal{G}_n^{\delta_0}(W^\ell)$ for all $j,k \geqslant 1$ and $\ell = 1,2$, without changing the bounds on sums over $\mathcal{G}_n^{\delta_0}(W^\ell)$. There is at most one $U_j^\ell$ and two $V_j^\ell$ per $W_i^\ell \in \mathcal{G}_n^{\delta_0}(W^\ell)$.

In this way we write $W^\ell = (\cup_j T^n U_j^\ell) \cup (\cup_i T^n V_i^\ell)$. Note that the images $T^n V_i^\ell$ of the unmatched pieces must be short while the images of the matched pieces $U_j^\ell$ may be long or short.

We have arranged a pairing of the pieces $U_j^\ell = G_{U_j^\ell}(I_j)$, $\ell = 1,2$, with the property:

$$\text{If } \; U_j^1 = \{(r, \varphi_{U_j^1}(r)) \mid r \in I_j\} \text{ then } U_j^2 = \{(r, \varphi_{U_j^2}(r)) \mid r \in I_j\}, \tag{3.5.5}$$

so that the point $x = (r, \varphi_{U_j^1}(r))$ is associated with the point $\bar{x} = (r, \varphi_{U_j^2}(r))$ by the vertical segment $\gamma_x \subset \{(r,s)\}_{s\in[-\pi/2,\pi/2]}$, for each $r \in I_j$.

Given $\psi_\ell$ on $W^\ell$ with $|\psi_\ell|_{C^\alpha(W^\ell)} \leqslant 1$ and $d(\psi_1,\psi_2) \leqslant \tilde{\varepsilon}$, we must estimate

$$\begin{aligned}\left|\int_{W^1} \mathcal{L}_g^n f \psi_1 \mathrm{d}m_{W_1} - \int_{W^2} \mathcal{L}_g^n f\psi_2 \mathrm{d}m_{W_2}\right| &\leqslant \sum_{l,i}\left|\int_{V_i^l} f\psi_l \circ T^n e^{S_n g}\mathrm{d}m\right| \\ &+ \sum_j \left|\int_{U_j^1} f\psi_1 \circ T^n e^{S_n g}\mathrm{d}m - \int_{U_j^2} f\psi_2 \circ T^n e^{S_n g}\mathrm{d}m\right|.\end{aligned} \tag{3.5.6}$$

We first estimate the differences of matched pieces $U_j^l$. The function $\phi_j = (\psi_1 \circ T^n e^{S_n g}) \circ G_{U_j^1} \circ G_{U_j^2}^{-1}$ is well defined on $U_j^2$, and we can estimate each difference by

$$\begin{aligned}\left|\int_{U_j^1} f\psi_1 \circ T^n e^{S_n g}\mathrm{d}m - \int_{U_j^2} f\psi_2 \circ T^n e^{S_n g}\mathrm{d}m\right| &\leqslant \left|\int_{U_j^1} f\psi_1 \circ T^n e^{S_n g}\mathrm{d}m - \int_{U_j^2} f\phi_j \mathrm{d}m\right| \\ &+ \left|\int_{U_j^2} f(\phi_j - \psi_2 \circ T^n e^{S_n g})\mathrm{d}m\right|.\end{aligned} \tag{3.5.7}$$

We bound the first term in equation (3.5.7) using the strong unstable norm. We have that $|G_{U_j^1} \circ G_{U_j^2}^{-1}|_{C^1} \leqslant C_g$, for some $C_g > 0$ due to the fact that each curve $U_j^l$ has uniformly bounded curvature and slopes bounded away from infinity. Thus $|\phi_j|_{C^\alpha(U_j^2)} \leqslant C C_g |\psi_1|_{C^\alpha(W^1)} |e^{S_n g}|_{C^\alpha(W^1)}$. Moreover, $d(\psi_1 \circ T^n e^{S_n g}, \phi_j) = |\psi_1 \circ T^n e^{S_n g} \circ G_{U_j^1} - \phi_j \circ G_{U_j^2}| = 0$ by definition of $\phi_j$. To complete the bound on the first term, we need the following estimate from [DZ11, Lemma 4.2]: There exists $C > 0$, independent of $W^1$ and $W^2$, such that

$$d_{\mathcal{W}^s}(U_j^1, U_j^2) \leqslant C\Lambda^{-n} n\tilde{\varepsilon} =: \varepsilon_1, \quad \forall j. \tag{3.5.8}$$

Then we apply the definition of the strong unstable norm with $\varepsilon_1$ instead of $\tilde{\varepsilon}$. Thus,

$$\sum_j \left| \int_{U_j^1} f\psi_1 \circ T^n e^{S_n g} \mathrm{d}m - \int_{U_j^2} f\phi_j \mathrm{d}m \right| \leqslant 2\delta_0^{-1} C C_g^2 |\log \varepsilon_1|^{-\zeta} ||f||_u \sum_{A \in \mathcal{M}_0^n} |e^{S_n g}|_{C^0(A)}, \tag{3.5.9}$$

where we used Lemmas 3.3.12 and 3.3.1(b) with $\gamma = 0$ since there is at most one matched piece $U_j^1$ corresponding to each component $W_i^1 \in \mathcal{G}_n^{\delta_0}(W^1)$ of $T^{-n}W^1$.

It remains to estimate the second term using the strong stable norm.

$$\left| \int_{U_j^2} f(\phi_j - \psi_2 \circ T^n e^{S_n g}) \mathrm{d}m \right| \leqslant ||f||_s |\log |U_j^2||^{-\gamma} |\phi_j - \psi_2 \circ T^n e^{S_n g}|_{C^\beta(U_j^2)}.$$

In order to estimate this last $C^\beta$-norm, we use that $|G_{U_j^2}|_{C^1} \leqslant C$ and $|G_{U_j^2}^{-1}|_{C^1} \leqslant C$.

$$\begin{aligned} |\phi_j - \psi_2 \circ T^n e^{S_n g}|_{C^\beta(U_j^2)} &\leqslant C|(\psi_1 \circ T^n e^{S_n g}) \circ G_{U_j^1} - (\psi_2 \circ T^n e^{S_n g}) \circ G_{U_j^2}|_{C^\beta(I_j)} \\ &\leqslant C|(\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2})(e^{S_n g} \circ G_{U_j^1}) \\ &\qquad + (\psi_2 \circ T^n \circ G_{U_j^2})(e^{S_n g} \circ G_{U_j^1} - e^{S_n g} \circ G_{U_j^2})|_{C^\beta(I_j)} \\ &\leqslant C|(\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2})|_{C^\beta(I_j)} \, |e^{S_n g}|_{C^0(U_j^1)} \\ &\qquad + C|\psi_2|_{C^0(U_j^2)} \, |e^{S_n g} \circ G_{U_j^1} - e^{S_n g} \circ G_{U_j^2}|_{C^\beta(I_j)}. \end{aligned} \tag{3.5.10}$$

It follows from [DZ11, Lemma 4.4] that

$$|\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)} \leqslant C\tilde{\varepsilon}^{\alpha - \beta}.$$

Now, we need to estimate $|e^{S_n g} \circ G_{U_j^1} - e^{S_n g} \circ G_{U_j^2}|_{C^\beta(I_j)}$. Since $d(T^i(G_{U_j^1}(r)), T^i(G_{U_j^2}(r))) \leqslant C\Lambda^{-(n-i)}\tilde{\varepsilon}$ for all $r \in I_j$ and $0 \leqslant i \leqslant n$, we get

$$\begin{aligned} |e^{S_n g} \circ G_{U_j^1}(r) - e^{S_n g} \circ G_{U_j^2}(r)| &= e^{S_n g(G_{U_j^1}(r))} |1 - e^{S_n g(G_{U_j^2}(r)) - S_n g(G_{U_j^1}(r))}| \\ &\leqslant 2|e^{S_n g}|_{C^0(U_j^1)} |S_n g(G_{U_j^2}(r)) - S_n g(G_{U_j^1}(r))| \\ &\leqslant 2C \frac{\Lambda^{\alpha_g}}{\Lambda^{\alpha_g} - 1} |g|_{C^{\alpha_g}} (C\tilde{\varepsilon})^{\alpha_g} |e^{S_n g}|_{C^0(U_j^1)} \end{aligned} \tag{3.5.11}$$

We estimate the $\beta$-Hölder constant in two ways. First, using (3.5.11) twice, we have for all $r$, $s \in I_j$ that

$$|e^{S_n g} \circ G_{U_j^1}(r) - e^{S_n g} \circ G_{U_j^2}(r) - e^{S_n g} \circ G_{U_j^1}(s) + e^{S_n g} \circ G_{U_j^2}(s)| \leqslant C\tilde{\varepsilon}^{\alpha_g} |e^{S_n g}|_{C^0(U_j^1)}.$$

On the other hand, using that $G_{U_j^\ell}(r)$ and $G_{U_j^\ell}(s)$ lie on the same stable curve,

$$\begin{aligned} &|e^{S_n g} \circ G_{U_j^1}(r) - e^{S_n g} \circ G_{U_j^2}(r) - e^{S_n g} \circ G_{U_j^1}(s) + e^{S_n g} \circ G_{U_j^2}(s)| \\ &\quad \leqslant |e^{S_n g} \circ G_{U_j^1}(r) - e^{S_n g} \circ G_{U_j^1}(s) + e^{S_n g} \circ G_{U_j^2}(s) - e^{S_n g} \circ G_{U_j^2}(r)| \\ &\quad \leqslant |e^{S_n g}|_{C^{\alpha_g}(U_j^1)} d(G_{U_j^1}(r), G_{U_j^1}(s))^{\alpha_g} + |e^{S_n g}|_{C^{\alpha_g}(U_j^2)} d(G_{U_j^2}(r), G_{U_j^2}(s))^{\alpha_g} \\ &\quad \leqslant C|e^{S_n g}|_{C^0(U_j^1)} |r - s|^{\alpha_g}. \end{aligned}$$

Thus, this quantity is bounded by the min of the two estimates. This min is maximal when the two upper bounds are equal, that is when $\tilde{\varepsilon} = C|r-s|$. Therefore, the $\beta$-Hölder constant satisfies

$$H^{\beta}_{I_j}(e^{S_n g} \circ G_{U^1_j} - e^{S_n g} \circ G_{U^2_j}) \leqslant C\tilde{\varepsilon}^{\alpha_g - \beta} |e^{S_n g}|_{C^0(U^1_j)} \,.$$

We therefore have proved that

$$|e^{S_n g} \circ G_{U^1_j} - e^{S_n g} \circ G_{U^2_j}|_{C^\beta(I_j)} \leqslant C\tilde{\varepsilon}^{\alpha_g - \beta} |e^{S_n g}|_{C^0(U^1_j)} \,.$$

Combining the above estimates inside (3.5.10), we finally have

$$|\phi_j - \psi_2 \circ T^n e^{S_n g}|_{C^\beta(U^2_j)} \leqslant C\tilde{\varepsilon}^{\alpha-\beta} |e^{S_n g}|_{C^0(U^1_j)} \,.$$

Summing over $j$ yields

$$\sum_j \left| \int_{U^2_j} f(\phi_j - \psi_2 \circ T^n e^{S_n g}) \mathrm{d}m \right| \leqslant C |\log \delta_0|^{-\gamma} ||f||_s \tilde{\varepsilon}^{\alpha-\beta} 2\delta_0^{-1} \sum_{A \in \mathcal{M}^n_0} |e^{S_n g}|_{C^0(A)},$$

where we used Lemma 3.3.1(b) with $\gamma = 0$ since there is at most one matched piece $U^l_j$ corresponding to each component $W^l_i \in \mathcal{G}^{\delta_0}_n(W^l)$ of $T^{-n}W^l$. Since $\delta_0 < 1$ is fixed, this completes the estimate on the second term of the matched pieces (originating from (3.5.6)).

We now turn to the estimate of the first sum in (3.5.6) concerning the unmatched pieces.

We say an unmatched curve $V^1_i$ is created at time $j$, $1 \leq j \leq n$, if $j$ is the first time that $T^{n-j}V^1_i$ is not part of a matched element of $\mathcal{G}^{\delta_0}_j(W^1)$. Indeed, there may be several curves $V^1_i$ (in principle exponentially many in $n-j$) such that $T^{n-j}V^1_i$ belongs to the same unmatched element of $\mathcal{G}^{\delta_0}_j(W^1)$. Define

$$\begin{aligned} A_{j,k} = \{ i \mid\ & V^1_i \text{ is created at time } j \\ & \text{and } T^{n-j}V^1_i \text{ belongs to the unmatched curve } W^1_k \subset T^{-j}W^1 \} \,. \end{aligned} \tag{3.5.12}$$

Due to the uniform hyperbolicity of $T$, and, again, uniform transversality of $\mathcal{S}_{-n}$ with the stable cone and of $C^s(x)$ with $C^u(x)$, we have $|W^1_k| \leqslant C\Lambda^{-j}\tilde{\varepsilon}$.

Recall that from Lemma 3.3.1(a) for $\bar{\gamma} = 0$, if for a certain time $q$, every element of $\mathcal{G}^{\delta_0}_q(W^1_k)$ have length less than $\delta_0/3$ – that is, if $\mathcal{G}^{\delta_0}_q(W^1_k) = \mathcal{I}^{\delta_0}_q(W^1_k)$ – then we have the subexponential growth

$$\sum_{V \in \mathcal{G}^{\delta_0}_q(W^1_k)} |e^{S_q g}|_{C^0(V)} \leqslant 2(Km+1)^{q/m} e^{q \sup g} \,. \tag{3.5.13}$$

We would like to establish a lower bound on the value of $q$ as a function of $j$.

More precisely, we want to find $q(j)$, as large as possible, so that

(a) $\mathcal{G}^{\delta_0}_{q(j)}(W^1_k) = \mathcal{I}^{\delta_0}_{q(j)}(W^1_k)$;

(b) $\dfrac{|\log|V||^{-\gamma}}{|\log \tilde{\varepsilon}|^{-\varsigma}} \leqslant 1$, for all $V \in \mathcal{I}^{\delta_0}_{q(j)}(W^1_k)$.

This is the content of the next two lemmas.

**Lemma 3.5.4.** *If $W \in \widehat{\mathcal{W}}^s$ is such that $\tilde{C}^2|W|^{2^{-kn_0 s_0}} < \delta_0/3$ for some $k \geqslant 1$, where $\tilde{C}$ is the constant from (3.3.2). Then $\mathcal{G}^{\delta_0}_{kn_0}(W) = \mathcal{I}^{\delta_0}_{kn_0}(W)$, and for all $1 \leqslant l \leqslant k$, and all $W_i \in \mathcal{G}^{\delta_0}_{ln_0}(W)$, $|W_i| \leqslant \tilde{C}^2|W|^{2^{-ln_0 s_0}}$.*

*Proof.* We prove the lemma by induction on $k$. We start with the case $k=1$. Let $1 \leqslant l \leqslant n_0$ and $W_i \in \mathcal{G}_l^{\delta_0}(W)$. Denote $V = T^l W_i \subset W$. Then, for all $0 \leqslant j \leqslant l$, $|T^j W_i| \leqslant \delta_0$. Decomposing $T^{-l}V = W_i$ as in the beginning of the proof of Lemma 3.3.1, we get that $|W_i| \leqslant \tilde{C}|W|^{2^{-n_0 s_0}}$, which is less than $\delta_0/3$ by assumption. Thus, $\mathcal{G}_l^{\delta_0}(W) = S_l^{\delta_0}(W)$ for each $0 \leqslant l \leqslant n_0$. Therefore $\mathcal{G}_{n_0}^{\delta_0}(W) = \mathcal{I}_{n_0}^{\delta_0}(W)$, with the claimed estimate.

Consider now the case $k>1$. Notice that, by construction, we have

$$\mathcal{G}_{(k+1)n_0}^{\delta_0}(W) = \bigcup_{W_i \in \mathcal{G}_{kn_0}^{\delta_0}(W)} \mathcal{G}_{n_0}^{\delta_0}(W).$$

Thus, we can apply the same method to estimate the length of an element $W_i \in \mathcal{G}_{kn_0+l}^{\delta_0}(W)$ from the length of its *parent* in $\mathcal{G}_{kn_0}^{\delta_0}(W)$, iterating the estimates in the same fashion as for (3.3.2). □

**Lemma 3.5.5.** *The above conditions (a) and (b) are satisfied for* $q(j) := \frac{(\gamma-\zeta)\log(j-j_0)}{\gamma s_0 \log 2} - 1$, *for all* $j \geqslant j_1$, *where* $j_1 > j_0 \geqslant 0$ *are constants (uniform in* $\tilde{\varepsilon}$ *and* $W_k^1$*). For* $j < j_1$, *set* $q(j) = 0$.

*Proof.* Since $|W_k^1| \leqslant C\tilde{\varepsilon}\Lambda^{-j}$ and using Lemma 3.5.4, the condition (a) will be satisfied whenever $\tilde{C}^2(C\tilde{\varepsilon}\Lambda^{-j})^{2^{-qs_0}} \leqslant \delta_0/3$.

Let $j_0$ be such that $C\Lambda^{-j_0} < 1$. Then (a) is satisfied whenever $\tilde{C}^2\Lambda^{-(j-j_0)2^{-qs_0}} \leqslant \delta_0/3$, that is

$$q \leqslant \frac{\log(j-j_0)}{s_0 \log 2} - C_2\,, \text{ with } C_2 := \frac{1}{s_0\log 2} \log \frac{\log \frac{3\tilde{C}^2}{\delta_0}}{\log \Lambda}. \tag{3.5.14}$$

Note that $C_2$ is uniform, and that the right-hand-side of (3.5.14) is larger than $q(j)$ for all $j$ large enough, say $j \geqslant j_1$.

Using the estimate from Lemma 3.5.4, condition (b) is satisfied whenever $q$ is such that $|\log \tilde{C}^2(C\tilde{\varepsilon}\Lambda^{-j})^{2^{-qs_0}}|^\gamma > |\log\tilde{\varepsilon}|^\zeta$. Now, we have that

$$|\log \tilde{C}^2(C\tilde{\varepsilon}\Lambda^{-j})^{2^{-qs_0}}| = |\log \tilde{C}^2 + 2^{-qs_0}\log(C\tilde{\varepsilon}\Lambda^{-j})| > \frac{1}{2}|2^{-qs_0}\log(C\tilde{\varepsilon}\Lambda^{-j})|,$$

whenever

$$q+1 \leqslant \frac{\log(j-j_0)}{s_0\log 2} + C_3\,, \text{ with } C_3 = \frac{1}{s_0 \log 2}\log\frac{\log\Lambda}{\log \tilde{C}^2} \tag{3.5.15}$$

Note that $C_3$ is uniform, and that the right-hand-side of (3.5.15) is larger than $q(j)$ for all $j$ large enough, say $j \geqslant j_1$ (up to increasing the value of $j_1$).

We thus have to prove that $|\log C\tilde{\varepsilon}\Lambda^{-j}|^\gamma > 2^{(q+1)s_0\gamma}|\log\tilde{\varepsilon}|^\zeta$ (which implies (b)). Notice that, from the definition of $q(j)$, we have $2^{(q(j)+1)s_0\gamma} \leqslant (j-j_0)^{\gamma-\zeta}$. We distinguish two cases.

Assume first that $(j-j_0)\log\Lambda \geqslant |\log\tilde{\varepsilon}|$. Therefore

$$\begin{aligned}
2^{(q(j)+1)\gamma}|\log\tilde{\varepsilon}|^\zeta &\leqslant (j-j_0)^{\gamma-\zeta}|\log\tilde{\varepsilon}|^\zeta \leqslant (j-j_0)^\gamma(\log\Lambda)^\zeta \leqslant ((j-j_0)\log\Lambda)^\gamma \\
&\leqslant ((j-j_0)\log\Lambda + |\log\tilde{\varepsilon}| + |\log C\Lambda^{-j_0}|)^\gamma \\
&\leqslant |-(j-j_0)\log\Lambda + \log\tilde{\varepsilon} + \log C\Lambda^{-j_0}|^\gamma \\
&\leqslant |\log C\tilde{\varepsilon}\Lambda^{-j}|^\gamma.
\end{aligned}$$

On the other hand, if $(j-j_0)\log\Lambda \leqslant |\log\tilde{\varepsilon}|$, then

$$\begin{aligned}
2^{(q(j)+1)\gamma}|\log\tilde{\varepsilon}|^{\zeta} &\leqslant (j-j_0)^{\gamma-\zeta}|\log\tilde{\varepsilon}|^{\zeta} \leqslant \frac{|\log\tilde{\varepsilon}|^{\gamma-\zeta}}{(\log\Lambda)^{\gamma-\zeta}}|\log\tilde{\varepsilon}|^{\zeta} \leqslant |\log\tilde{\varepsilon}|^{\gamma} \\
&\leqslant ((j-j_0)\log\Lambda + |\log\tilde{\varepsilon}| + |\log C\Lambda^{-j_0}|)^{\gamma} \\
&\leqslant |-(j-j_0)\log\Lambda + \log\tilde{\varepsilon} + \log C\Lambda^{-j_0}|^{\gamma} \\
&\leqslant |\log C\tilde{\varepsilon}\Lambda^{-j}|^{\gamma}.
\end{aligned}$$

Thus, the choice $q(j)$ satisfies (a) and (b) for all $j \geqslant j_1$. □

Next, recalling the $A_{j,k}$ from (3.5.12), we estimate[15] over the unmatched pieces $V_i^l$ in (3.5.6), using the strong stable norm. Since cases $l=1$ and $l=2$ are similar here, we only deal with the case $l=1$.

$$\begin{aligned}
&\sum_{V_i^1}\left|\int_{V_i^1} f\psi_1\circ T^n e^{S_n g}\mathrm{d}m_{V_i^1}\right| = \sum_{j=1}^{n}\sum_{k}\sum_{i\in A_{j,k}}\left|\int_{T^{n-j}V_i^1}(\mathcal{L}_g^{n-j}f)\psi_1\circ T^j e^{S_j g}\right| \\
&\leqslant \sum_{j=1}^{n}\sum_{k}\sum_{V_l\in\mathcal{G}^{\delta_0}_{q(j)}(W_k^1)}\left|\int_{V_l}(\mathcal{L}_g^{n-j-q(j)}f)\psi_1\circ T^{j+q(j)}e^{S_{j+q(j)}g}\right| \\
&\leqslant \sum_{j=1}^{n}\sum_{k}\sum_{V_l\in\mathcal{G}^{\delta_0}_{q(j)}(W_k^1)} ||\mathcal{L}_g^{n-j-q(j)}f||_s C|\log|V_l||^{-\gamma}|\psi_1\circ T^{j+q(j)}|_{C^\beta(V_l)}|e^{S_{j+q(j)}g}|_{C^\beta(V_l)} \\
&\leqslant C||f||_s\sum_{j=1}^{n}\frac{C}{c_1\delta_0}e^{(n-j-q(j))P_*(T,g)}|\log\tilde{\varepsilon}|^{-\zeta}\sum_{k}\sum_{V_l\in\mathcal{G}^{\delta_0}_{q(j)}(W_k^1)}|e^{S_{j+q(j)}g}|_{C^\beta(V_l)} \\
&\leqslant \frac{C}{c_1\delta_0}||f||_s\sum_{j=1}^{n}e^{(n-j-q(j))P_*(T,g)}|\log\tilde{\varepsilon}|^{-\zeta}\sum_{W_k^1\subset T^{-j}W^1}\sum_{V_l\in\mathcal{G}^{\delta_0}_{q(j)}(W_k^1)}|e^{S_j g\circ T^{q(j)}+S_{q(j)}g}|_{C^\beta(V_l)} \\
&\leqslant \frac{C}{c_1\delta_0}||f||_s\sum_{j=1}^{n}e^{(n-j-q(j))P_*(T,g)}|\log\tilde{\varepsilon}|^{-\zeta}\sum_{W_k^1\subset T^{-j}W^1}|e^{S_j g}|_{C^0(W_k^1)}\sum_{V_l\in\mathcal{G}^{\delta_0}_{q(j)}(W_k^1)}|e^{S_{q(j)}g}|_{C^0(V_l)} \\
&\leqslant \frac{C}{c_1\delta_0}||f||_s\sum_{j=1}^{n}e^{(n-j-q(j))P_*(T,g)}|\log\tilde{\varepsilon}|^{-\zeta}\frac{4C}{c_1\delta_0}e^{jP_*(T,g)}e^{q(j)\sup g}(Km+1)^{q(j)/m} \\
&\leqslant \frac{C}{(c_1\delta_0)^2}||f||_s|\log\tilde{\varepsilon}|^{-\zeta}e^{nP_*(T,g)}\sum_{j=1}^{n}e^{-q(j)(P_*(T,g)-\sup g-\frac{1}{m}\log(Km+1))}.
\end{aligned}$$

Now, for $\tilde{\varepsilon}>0$, fixed, since we assume that $P_*(T,g)-\sup g > s_0\log 2$, we can chose $m$ large enough and $\zeta$ small enough such that $\varepsilon_1 := P_*(T,g)-\sup g-\frac{1}{m}\log(Km+1)-\frac{\gamma}{\gamma-\zeta}s_0\log 2 > 0$. By definition of $q(j)$, we obtain that

$$\begin{aligned}
\sum_{j=j_1}^{n}e^{-q(j)(P_*(T,g)-\sup g-\frac{1}{m}\log(Km+1))} &= \sum_{j=j_1}^{n}e^{-\frac{(\gamma-\zeta)\log(j-j_0)}{\gamma s_0\log 2}(\varepsilon_1+\frac{\gamma}{\gamma-\zeta}s_0\log 2)} \\
&= \sum_{j=j_1}^{n}(j-j_0)^{-1-\frac{\gamma-\zeta}{\gamma s_0\log 2}\varepsilon_1},
\end{aligned}$$

is bounded. The bound (3.5.3) then follows by combining all the above estimates into (3.5.6) and taking the appropriate suprema. □

15. For the $4^{\text{th}}$ and $6^{\text{th}}$ inequalities, we use Proposition 3.3.10. Here again, $P_*(T,g)$ can be replaced by $P_*(T,g)+\varepsilon$ up to a larger multiplicative constant.

*Remark* 3.5.6. In the case $g = -h_{\text{top}}(\phi_1)\tau$, the assumption $P_*(T, g) - \sup g > s_0 \log 2$ in Proposition 3.5.1 is implied by the condition $h_{\text{top}}(\phi_1)\tau_{\min} > s_0 \log 2$, which is itself implied by $\tau_{\min} h_{\mu_{\text{SRB}}}(T)/\mu_{\text{SRB}}(\tau) > s_0 \log 2$ thanks to the Abramov formula. This latter condition is satisfied for billiards studied by Baras and Gaspard [GB95] and by Garrido [Gar97], as long as $\tau_{\min}$ is *not too small*.

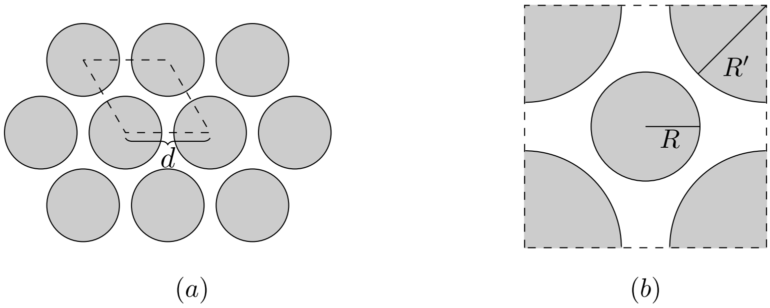



**Figure 3.1** – (a) The Sinai billiard on a triangular lattice studied in [GB95] with angle $\pi/3$, scatterer of radius 1, and distance $d$ between the centers of adjacent scatterers. (b) The Sinai billiard on a square lattice with scatterers of radius $R < R'$ studied in [Gar97]. The boundary of a single cell is indicated by dashed lines in both tables.

Indeed, Garrido [Gar97] studied the Sinai billiard corresponding to the periodic Lorentz gas with two scatterers of radius $R < R'$ on the unit square lattice (Figure 3.1(b)). Setting $R' = 0.4$, Garrido computed $h_{\mu_{\text{SRB}}}(T)$ and $\mu_{\text{SRB}}(\tau)$ for about 20 values of $R$ ranging from $R = 0.1$ (when the horizon becomes infinite) to $R = \frac{\sqrt{2}}{2} - 0.4$ (when the scatterers touch: $\tau_{\min} = 0$). According to [BD20, § 2.4], in those examples we can always find $\varphi_0$ and $n_0$ such that $s_0 \leqslant \frac{1}{2}$. Furthermore, $\tau_{\min} = \frac{\sqrt{2}}{2} - 0.4 - R$. Now, for $R = 0.1^+$, we find that

$$\tau_{\min} h_{\mu_{\text{SRB}}}(T)/\mu_{\text{SRB}}(\tau) \geqslant (\tfrac{\sqrt{2}}{2} - 0.5)\tfrac{1.7}{0.5} \geqslant 0.7 > \tfrac{1}{2}\log 2 \geqslant s_0 \log 2\,,$$

and for $R = 0.2$, we find that

$$\tau_{\min} h_{\mu_{\text{SRB}}}(T)/\mu_{\text{SRB}}(\tau) \geqslant (\tfrac{\sqrt{2}}{2} - 0.6)\tfrac{1.4}{0.3} \geqslant 0.48 > \tfrac{1}{2}\log 2 \geqslant s_0 \log 2\,.$$

Since for $R \in (0.1, 0.2]$, $R \mapsto \tau_{\min}(R)$ is a linear function, and according to Garrido Figures 6 and 8, $R \mapsto \mu_{\text{SRB}}(\tau)(R)$ is well approximated by an affine function and $R \mapsto h_{\mu_{\text{SRB}}}(T)(R)$ is lower bounded by an affine function joining the values at $R = 0.1$ and 0.2, it appears that the condition $\tau_{\min} h_{\mu_{\text{SRB}}}(T)/\mu_{\text{SRB}}(\tau) > s_0 \log 2$ is satisfied for all $R \in (0.1, 0.2]$.

Baras and Gaspard studied the Sinai billiard corresponding to the Lorentz gas with disks of radius 1 centered in a triangular lattice (Figure 3.1(a)). The distance $d$ between points on the lattice is varied from $d = 2$ (when the scatterers touch: $\tau_{\min} = 0$) to $d = 4/\sqrt{3}$ (when the horizon becomes infinite). We have that $\tau_{\min} = d - 2$ and, still according to [BD20, § 2.4], in those examples we can always find $\varphi_0$ and $n_0$ such that $s_0 \leqslant \frac{1}{2}$. The computed values are the average Lyapunov exponent of the billiard flows given in [GB95], provide a lower bound directly on $h_{\mu_{\text{SRB}}}(T)/\mu_{\text{SRB}}(\tau)$. For $d = 0.2$, we find

$$\tau_{\min} h_{\mu_{\text{SRB}}}(T)/\mu_{\text{SRB}}(\tau) \geqslant (\tfrac{4}{\sqrt{3}} - 2)\,1.8 \geqslant 0.55 > \tfrac{1}{2}\log 2 \geqslant s_0 \log 2\,.$$

The condition $h_{\text{top}}(\phi_1)\tau_{\min} > s_0 \log 2$ is a little bit more restrictive than the one used by Baladi and Demers in [BD20] since, by the Abramov formula, $h_* = h_{\text{top}}(\phi_1)\mu_*(\tau) \geqslant$

$h_{\text{top}}(\phi_1)\tau_{\min}$. (Also, we do not know any example of billiard for which the condition $h_* > s_0 \log 2$ is not satisfied.)

We now turn to the condition SSP.1. Unfortunately, we don't know any billiard table such that the potential $g = -h_{\text{top}}(\phi_1)\tau$ satisfies a *simple* condition implying SSP.1. By *simple*, we mean a sufficient condition that does not involve topological entropies, since they are notoriously hard to estimate numerically. First, recall from Lemmas 3.3.3 and 3.3.4 that $\log \Lambda > h_{\text{top}}(\phi_1)(\tau_{\max} - \tau_{\min})$ implies SSP.1. Remark that since $g$ and $\frac{1}{n}S_n g$ are cohomologous, they would give rise to the same equilibrium states. It is then advantageous to work with the Birkhoff average instead of $g$ because $\max \frac{1}{n}S_n g \leqslant \tau_{\max}$ and $\min \frac{1}{n}S_n g = \tau_{\min}$ (notice that $\tau_{\min}$ is achieved on an orbit of period 2). Now, taking advantage of the Abramov formula and of the variational principle, we get that $\max \frac{1}{n}S_n g < 2\tau_{\min}$ implies $h_* > h_{\text{top}}(\phi_1)(\max \frac{1}{n}S_n g - \tau_{\min})$ (recall that $h_* > \log \Lambda$ is the topological entropy of $T$, as defined in [BD20]). The condition $\max \frac{1}{n}S_n g < 2\tau_{\min}$ involves quantities that are easy to estimate numerically, however, we don't know any billiard table satisfying this condition.

We now deduce the bounds of Theorem 3.5.3 from the rate of growth of stable curves proved in Proposition 3.3.7.

*Proof of Theorem 3.5.3.* To prove this lower bound on $|\mathcal{L}_g^n 1|_w$, recall the choice of $\delta_1 > 0$ from Lemma 3.3.3 for $\varepsilon = 1/4$ (thus satisfying (3.3.4)). Let $W \in \mathcal{W}^s$ with $|W| \geqslant \delta_1/3$ and set the test function $\psi \equiv 1$. For $n \geqslant n_1$,

$$\int_W \mathcal{L}_g^n 1 \mathrm{d}m_W = \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} \int_{W_i} e^{S_n g} \mathrm{d}m_{W_i} \geqslant \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} \frac{\delta_1}{2} \inf_{W_i} e^{S_n g} \geqslant \frac{\delta_1}{2} C^{-1} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} \sup_{W_i} e^{S_n g},$$

where we used Lemma 3.2.3 for the second inequality, since for each $W_i \in \mathcal{G}_n^{\delta_1}(W)$ there exists $A \in \mathcal{M}_0^n$ such that $W_i \subset A$ and

$$\sup_{W_i} e^{S_n g} \leqslant \sup_A e^{S_n g} \leqslant C \inf_A e^{S_n g} \leqslant C \inf_{W_i} e^{S_n g}.$$

We can now use Proposition 3.3.7 to get

$$\int_W \mathcal{L}_g^n 1 \mathrm{d}m_W \geqslant \frac{\delta_1}{2C} c_0 \sum_{A \in \mathcal{M}_{-n}^0} |e^{S_n^{-1} g}|_{C^0(A)} \geqslant \frac{\delta_1}{2C} c_0 e^{nP_*(T,g)}. \tag{3.5.16}$$

Thus

$$||\mathcal{L}_g^n 1||_s \geqslant |\mathcal{L}_g^n 1|_w \geqslant \frac{\delta_1}{2} c_0 e^{nP_*(T,g)}.$$

Letting $n$ tend to infinity, one obtains $\lim_{n\to\infty} ||\mathcal{L}_g^n 1||_{\mathcal{B}}^{1/n} \geqslant e^{P_*(T,g)}$. □

## 3.6 The measure $\mu_g$

This section is devoted to the construction, the properties and the uniqueness of an equilibrium state $\mu_g$ for $T$, associated to a potential $g$.

We will assume throughout that $g$ is a $(\mathcal{M}_0^1, \alpha_g)$-Hölder potential such that $P_*(T,g) - \sup g > s_0 \log 2$ and that the conditions SSP.1 and SSP.2 are satisfied.

### 3.6.1 Construction of the measure $\mu_g$ – Measure of Singular Sets

In this section, we construct a $T$-invariant probability measure $\mu_g$ on $M$ by combining in (3.6.1) a maximal eigenvector of $\mathcal{L}_g$ on $\mathcal{B}$ and a maximal eigenvector of its dual, obtained in Proposition 3.6.1. In addition, the information on these left and right eigenvectors will give Lemma 3.6.2 and Corollary 3.6.3, which imply that $\mu_g$ is $T$-adapted.

We first show that such maximal eigenvectors exist and are in fact nonnegative Radon measures – that is, elements of the dual of $C^0(M)$.

**Proposition 3.6.1.** *If $g$ is a $(\mathcal{M}_0^1, \alpha_g)$-Hölder continuous potential such that $P_*(T,g) - \sup g > s_0 \log 2$ and $\log \Lambda > \sup g - \inf g$, then there exist $\nu \in \mathcal{B}_w$ and $\tilde{\nu} \in \mathcal{B}_w^*$ such that $\mathcal{L}_g \nu = e^{P_*(T,g)}\nu$ and $\mathcal{L}_g^* \tilde{\nu} = e^{P_*(T,g)}\tilde{\nu}$. In addition, $\nu$ and $\tilde{\nu}$ take nonnegative values on nonnegative $C^1$ functions on $M$ and are thus nonnegative Radon measures. Finally, $\tilde{\nu}(\nu) \neq 0$ and $||\nu||_u \leqslant \overline{C}$.*

It is easy to see that $|f\varphi|_w \leq |\varphi|_{C^1}|f|_w$ (use $|\varphi\psi|_{C^\alpha(W)} \leq |\varphi|_{C^1}|\psi|_{C^\alpha(W)}$). Clearly, if $f \in C^1$ and $\varphi \in C^1$ then $f\varphi \in C^1$. Therefore, if $P_*(T,g) - \sup g > s_0 \log 2$ and both SSP conditions are satisfied, a bounded linear map $\mu_g$ from $C^1(M)$ to $\mathbb{C}$ can be defined by taking $\nu$ and $\tilde{\nu}$ from Proposition 3.6.1 and setting

$$\mu_g(\varphi) = \frac{\tilde{\nu}(\varphi\nu)}{\tilde{\nu}(\nu)} . \tag{3.6.1}$$

This map is nonnegative for all nonnegative $\varphi$ and thus defines a nonnegative measure $\mu_g \in (C^0(M))^*$, with $\mu_g(1) = 1$. Clearly, $\mu_g$ is a $T$ invariant probability measure since for every $\varphi \in C^1$ we have

$$\tilde{\nu}(\varphi\nu) = e^{-P_*(T,g)}\tilde{\nu}(\varphi\mathcal{L}_g(\nu)) = e^{-P_*(T,g)}\tilde{\nu}(\mathcal{L}_g((\varphi \circ T)\nu)) = \tilde{\nu}((\varphi \circ T)\nu) .$$

*Proof.* Let 1 denote the constant function equal to 1 on $M$. We will take this as a seed in our construction of a maximal eigenvector. By Theorem 3.5.3, we see that $\|\mathcal{L}_g^n 1\|_{\mathcal{B}} \geqslant \|\mathcal{L}_g^n 1\|_s \geqslant |\mathcal{L}_g^n 1|_w \geqslant C e^{nP_*(T,g)}$. Now consider

$$\nu_n := \frac{1}{n}\sum_{k=0}^{n-1} e^{-kP_*(T,g)}\mathcal{L}_g^k 1, \quad n \geqslant 1. \tag{3.6.2}$$

By construction, the $\nu_n$ are nonnegative, and thus can be extended into Radon measures. By Proposition 3.5.1, they satisfy $||\nu_n||_{\mathcal{B}} \leqslant \overline{C}$, so using the relative compactness of $\mathcal{B}$ in $\mathcal{B}_w$ ( [BD20, Proposition 6.1]), we extract a subsequence $(n_j)$ such that $\lim_j \nu_{n_j} = \nu$ is a nonnegative Radon measure, and the convergence is in $\mathcal{B}_w$. Since $\mathcal{L}_g$ is continuous on $\mathcal{B}_w$, we may write,

$$\begin{aligned}
\mathcal{L}_g \nu &= \lim_{j\to\infty} \frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kP_*(T,g)}\mathcal{L}_g^{k+1} 1 \\
&= \lim_{j\to\infty} \frac{e^{P_*(T,g)}}{n_j}\sum_{k=0}^{n_j-1} e^{-kP_*(T,g)}\mathcal{L}_g^k 1 - \frac{1}{n_j}e^{P_*(T,g)}1 + \frac{1}{n_j}e^{(n_j-1)P_*(T,g)}\mathcal{L}_g^{n_j}1 \\
&= e^{P_*(T,g)}\nu,
\end{aligned}$$

where we used that the second and third terms go to 0 (in the $\mathcal{B}$-norm). We thus obtain a nonnegative measure $\nu \in \mathcal{B}_w$ such that $\mathcal{L}_g \nu = e^{P_*(T,g)}\nu$.

Although $\nu$ is not a priori an element of $\mathcal{B}$, it does inherit bounds on the unstable norm from the sequence $\nu_n$. The convergence of $(\nu_{n_j})$ to $\nu$ in $\mathcal{B}_w$ implies that

$$\lim_{j\to\infty} \sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in C^\alpha(W)\\ |\psi|_{C^\alpha(W)}\leqslant 1}} \left(\int_W \nu\psi\, dm_W - \int_W \nu_{n_j}\psi\, dm_W\right) = 0\,.$$

Since $||\nu_{n_j}||_u \leqslant \overline{C}$, it follows that $||\nu||_u \leqslant \overline{C}$, as claimed.

Next, recalling the bound $|\int f d\mu_{\text{SRB}}| \leqslant \hat{C}|f|_w$ from [BD20, Proposition 4.2], setting $d\mu_{\text{SRB}} \in (\mathcal{B}_w)^*$ to be the functional defined on $C^1(M) \subset \mathcal{B}_w$ by $d\mu_{\text{SRB}}(f) = \int f d\mu_{\text{SRB}}$ and extended by density, we define

$$\tilde{\nu}_n := \frac{1}{n}\sum_{k=0}^{n-1} e^{-kP_*(T,g)}(\mathcal{L}_g^*)^k(d\mu_{\text{SRB}}). \tag{3.6.3}$$

Then, we have $|\tilde{\nu}_n(f)| \leqslant C|f|_w$ for all $n$ and all $f \in \mathcal{B}_w$. So $\tilde{\nu}_n$ is bounded in $(\mathcal{B}_w)^* \subset \mathcal{B}^*$. By compactness of the embedding ( [BD20, Proposition 6.1]), we can find a subsequence $\tilde{\nu}_{\tilde{n}_j}$ converging to $\tilde{\nu} \in \mathcal{B}^*$. By the argument above, we have $\mathcal{L}_g^*\tilde{\nu} = e^{P_*(T,g)}\tilde{\nu}$.

We next check that $\tilde{\nu}$, which in principle lies in the dual of $\mathcal{B}$, is in fact an element of $(\mathcal{B}_w)^*$. For this, it suffices to find $\tilde{C} < \infty$ so that for any $f \in \mathcal{B}$ we have

$$\tilde{\nu}(f) \leqslant \tilde{C}|f|_w. \tag{3.6.4}$$

Now, for $f \in \mathcal{B}$ and any $n_j \geqslant 1$, we have

$$|\tilde{\nu}(f)| \leqslant |(\tilde{\nu}_{n_j} - \tilde{\nu})(f)| + |\tilde{\nu}_{n_j}(f)| \leqslant |(\tilde{\nu}_{n_j} - \tilde{\nu})(f)| + |f|_w.$$

Since $\tilde{\nu}_{\tilde{n}_j} \to \tilde{\nu}$ in $\mathcal{B}^*$, we conclude $|\tilde{\nu}(f)| \leqslant |f|_w$ for all $f \in \mathcal{B}$. Since $\mathcal{B}$ is dense in $\mathcal{B}_w$, by [RS80, Thm I.7] $\tilde{\nu}$ extends uniquely to a bounded linear functional on $\mathcal{B}_w$ satisfying (3.6.4). It only remains to prove that $\tilde{\nu}(\nu) > 0$.

Let $(n_j)$ (resp. $(\tilde{n}_j)$) denote the subsequence such that $\nu = \lim_j \nu_{n_j}$ (resp. $\tilde{\nu} = \lim_j \tilde{\nu}_{\tilde{n}_j}$). Since $\tilde{\nu}$ is continuous on $\mathcal{B}_w$, we have on the one hand

$$\tilde{\nu}(\nu) = \lim_{j\to\infty}\tilde{\nu}(\nu_{n_j}) = \lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kP_*(T,g)}\tilde{\nu}(\mathcal{L}_g^k 1) = \lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\tilde{\nu}(1) = \tilde{\nu}(1),$$

where we have used that $\tilde{\nu}$ is an eigenvector of $\mathcal{L}_g^*$. On the other hand,

$$\tilde{\nu}(1) = \lim_{j\to\infty}\frac{1}{\tilde{n}_j}\sum_{k=0}^{\tilde{n}_j-1} e^{-kP_*(T,g)}(\mathcal{L}_g^*)^k d\mu_{\text{SRB}}(1) = \lim_{j\to\infty}\frac{1}{\tilde{n}_j}\sum_{k=0}^{\tilde{n}_j-1} e^{-kP_*(T,g)}\int \mathcal{L}_g^k 1\, d\mu_{\text{SRB}}.$$

Next, we disintegrate $d\mu_{\text{SRB}}$ as in the proof of [BD20, Lemma 4.4] into conditional measures $\mu_{\text{SRB}}^{W_\xi}$ on maximal homogeneous stable manifolds $W_\xi \in \mathcal{W}_{\mathbb{H}}^s$ and a factor measure $d\hat{\mu}_{\text{SRB}}(\xi)$ on the index set $\Xi$ of stable manifolds. Recall that $\mu_{\text{SRB}}^{W_\xi} = |W_\xi|^{-1}\rho_\xi dm_W$, where $\rho_\xi$ is uniformly log-Hölder continuous so that

$$0 < c_\rho \leqslant \inf_{\xi\in\Xi}\inf_{W_\xi}\rho_\xi \leqslant \sup_{\xi\in\Xi}|\rho_\xi|_{C^\alpha(W_\xi)} \leqslant C_\rho < \infty. \tag{3.6.5}$$

Let $\Xi^{\delta_1}$ denote those $\xi \in \Xi$ such that $|W_\xi| \geqslant \delta_1/3$ and note that $\hat{\mu}_{\text{SRB}}(\Xi^{\delta_1}) > 0$. Then, disintegrating as usual, we get by (3.5.16) for $k \geqslant n_1$,

$$\begin{aligned}\int \mathcal{L}_g^k 1 \mathrm{d}\mu_{\text{SRB}} &= \int_\Xi \int_{W_\xi} \mathcal{L}_g^k 1 \rho_\xi |W_\xi|^{-1} \mathrm{d}m_{W_\xi} \mathrm{d}\hat{\mu}_{\text{SRB}}(\xi) \\ &\geqslant \int_{\Xi^{\delta_1}} \int_{W_\xi} \mathcal{L}_g^k 1 \mathrm{d}m_{W_\xi} c_\rho 3\delta_1^{-1} \mathrm{d}\hat{\mu}_{\text{SRB}}(\xi) \geqslant c_\rho \frac{3c_0}{2C} e^{kP_*(T,g)} \hat{\mu}_{\text{SRB}}(\Xi^{\delta_1}) > 0.\end{aligned}$$

Thus $\tilde{\nu}(\nu) = \tilde{\nu}(1) \geqslant c_\rho \frac{3c_0}{2C} \hat{\mu}_{\text{SRB}}(\Xi^{\delta_1}) > 0$ as required. □

**Lemma 3.6.2.** *For any $\gamma > 0$ such that $2^{s_0\gamma} < e^{P_*(T,g) - \sup g}$ and any $k \in \mathbb{Z}$ there exists $C_k > 0$ such that*

$$\mu_g(\mathcal{N}_\varepsilon(\mathcal{S}_k)) \leqslant C_k |\log \varepsilon|^{-\gamma}, \quad \forall \varepsilon > 0. \tag{3.6.6}$$

*In particular, for any $p > 1/\gamma$ (one can choose $p < 1$ for $\gamma > 1$), $\eta > 0$, and $k \in \mathbb{Z}$, for $\mu_g$-almost every $x \in M$, there exists $C > 0$ such that*

$$d(T^n x, \mathcal{S}_k) \geqslant C e^{-\eta n^p}, \quad \forall n \geqslant 0. \tag{3.6.7}$$

*Proof.* First, for each $k \geqslant 0$, we claim that there exists $C_k > 0$ such that for all $\varepsilon > 0$,

$$|\nu(\mathcal{N}_\varepsilon(\mathcal{S}_k))| \leqslant C |1_{k,\varepsilon} \nu|_w \leqslant C_k |\log \varepsilon|^{-\gamma}, \tag{3.6.8}$$

where $1_{k,\varepsilon} = \mathbb{1}_{\mathcal{N}_\varepsilon(\mathcal{S}_k)}$. The proof of the first inequality in (3.6.8) is formally the same as in the proof of [BD20, Lemma 7.3].

We now prove the second inequality in (3.6.8). Let $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \leqslant 1$. Due to the uniform transversality of curves in $\mathcal{S}_{-k}$ with the stable cone, the intersection $W \cap \mathcal{N}_\varepsilon(\mathcal{S}_{-k})$ can be expressed as a finite union with cardinality bounded by a constant $A_k$ (depending only on $\mathcal{S}_{-k}$) of stable manifolds $W_i \in \mathcal{W}^s$, of lengths at most $C\varepsilon$. Therefore, for any $f \in C^1(M)$,

$$\int_{W_\xi} f 1_{k,\varepsilon} \psi \, \mathrm{d}m_W = \sum_i \int_{W_i} f\psi \, \mathrm{d}m_{W_i} \leqslant \sum_i |f|_w |\psi|_{C^\alpha(W_i)} \leqslant C A_k |f|_w.$$

It follows that $|1_{k,\varepsilon} f|_w \leqslant A_k |f|_w$ for all $f \in \mathcal{B}_w$. Similarly, we have $|1_{k,\varepsilon} f|_w \leqslant A_k ||f||_s |\log \varepsilon|^{-\gamma}$ for all $f \in \mathcal{B}$. Now, recalling $\nu_n$, we estimate,

$$|1_{k,\varepsilon} \nu|_w \leqslant |1_{k,\varepsilon}(\nu - \nu_n)|_w + |1_{k,\varepsilon} \nu_n|_w \leqslant A_k |\nu - \nu_n|_w + C_k' |\log \varepsilon|^{-\gamma} ||\nu_n||_{\mathcal{B}}.$$

Since $||\nu_n||_{\mathcal{B}} \leqslant \overline{C}$ for all $n \geqslant 1$, we take the limit as $n \to \infty$ to conclude that $|1_{k,\varepsilon} \nu|_w \leqslant C_k |\log \varepsilon|^{-\gamma}$, concluding the proof of (3.6.8).

Next, applying (3.6.4), we have

$$\tilde{\nu}(\nu)\, \mu_g(\mathcal{N}_\varepsilon(\mathcal{S}_{-k})) = \tilde{\nu}(1_{k,\varepsilon} \nu) \leqslant \tilde{C} |1_{k,\varepsilon} \nu|_w \leqslant \tilde{C} C_k |\log \varepsilon|^{-\gamma} \quad \forall k \geqslant 0\,.$$

To obtain the analogous bound for $\mathcal{N}_\varepsilon(\mathcal{S}_k)$, for $k > 0$, we use the invariance of $\mu_g$. It follows from [CM06, Exercice 4.50] that $T(\mathcal{N}_\varepsilon(\mathcal{S}_1)) \subset \mathcal{N}_{C\varepsilon^{1/2}}(\mathcal{S}_{-1})$. Thus,

$$\mu_g(\mathcal{N}_\varepsilon(\mathcal{S}_1)) \leqslant \mu_g(\mathcal{N}_{C\varepsilon^{1/2}}(\mathcal{S}_{-1})) \leqslant C_1 |\log C\varepsilon^{1/2}|^{-\gamma} \leqslant C_1' |\log \varepsilon|^{-\gamma}\,.$$

The estimate for $\mathcal{N}_\varepsilon(\mathcal{S}_k)$, for $k \geqslant 2$, follows similarly since $T^k \mathcal{S}_k = \mathcal{S}_{-k}$.

Finally, fix $\eta > 0$, $k \in \mathbb{Z}$ and $p > 1/\gamma$. Since

$$\sum_{n\geqslant 0} \mu_g(\mathcal{N}_{e^{-\eta n^p}}(\mathcal{S}_k)) \leqslant \tilde{C} C_k \eta^{-\gamma} \sum_{n\geqslant 1} n^{-p\gamma} < \infty\,, \tag{3.6.9}$$

by the Borel–Cantelli Lemma, the positive orbit of $\mu_g$-almost every $x \in M$ visits $\mathcal{N}_{e^{-\eta n^p}}(\mathcal{S}_k)$ only finitely many times, and the last part of the lemma follows. □

**Corollary 3.6.3.** *a) For any $\gamma > 0$ so that $P_*(T, g) - \sup g > \gamma s_0 \log 2$ and any $C^1$ curve $S$ uniformly transverse to the stable cone, there exists $C > 0$ such that $\nu(\mathcal{N}_\varepsilon(S)) \leqslant C|\log\varepsilon|^{-\gamma}$ and $\mu_g(\mathcal{N}_\varepsilon(S)) \leqslant C|\log\varepsilon|^{-\gamma}$ for all $\varepsilon > 0$.*

*b) The measures $\nu$ and $\mu_g$ have no atoms, and $\mu_g(W) = 0$ for all $W \in \mathcal{W}^s$ and $W \in \mathcal{W}^u$.*

*c) The measure $\mu_g$ is adapted: $\int |\log d(x, \mathcal{S}_{\pm 1})|\, \mathrm{d}\mu_g < \infty$.*

*d) $\mu_g$-almost every point in $M$ has a stable and unstable manifold of positive length.*

*Proof.* The proof is identical to the one of [BD20, Corollary 7.4], where $\mu_*$ should be replace by $\mu_g$. □

### 3.6.2 $\nu$-Almost Everywhere Positive Length of Unstable Manifolds

In this section, we establish almost everywhere positive length of unstable manifolds in the sense of the measure $\nu$ – the maximal eigenvector of $\mathcal{L}_g$ in $\mathcal{B}_w$, extended into a measure since it is nonnegative distribution. To do so, we will view elements of $\mathcal{B}_w$ as *leafwise measure* (Definition 3.6.4). Indeed, in Lemma 3.6.6, we make a connection between the disintegration of $\nu$ as a measure, and the family of leafwise measures on the set of stable manifolds $\mathcal{W}^s$.

**Definition 3.6.4** (Leafwise distribution and leafwise measure)**.** *For $f \in C^1(M)$ and $W \in \mathcal{W}^s$, the map defined on $C^\alpha(W)$ by*

$$\psi \mapsto \int_W f\psi\, \mathrm{d}m_W,$$

*can be viewed as a distribution of order $\alpha$ on $W$. Since $|\int_W f\psi\, \mathrm{d}m_W| \leqslant |f|_w |\psi|_{C^\alpha(W)}$, we can extend the map sending $f \in C^1(M)$ to this distribution of order $\alpha$, to $f \in \mathcal{B}_w$. We denote this extension by $\int_W f\psi\, \mathrm{d}m_W$ or $\int_W \psi f$, and we call the corresponding family of distributions $(f, W)_{W\in\mathcal{W}^s}$ the leafwise distribution associated to $f \in \mathcal{B}_w$.*
*Note that if $\int_W f\psi \mathrm{d}m_W \geqslant 0$ for all $\psi \geqslant 0$, then the leafwise distribution on $W$ can be extended into a bounded linear functional on $C^0(W)$, or in other words, a Radon measure. If this holds for all $W \in \mathcal{W}^s$, the leafwise distribution is called a leafwise measure.*

**Lemma 3.6.5** (Almost Everywhere Positive Length of Unstable Manifolds, for $\nu$)**.** *For $\nu$-almost every $x \in M$ the stable and unstable manifolds have positive length. Moreover, viewing $\nu$ as a leafwise measure, for every $W \in \mathcal{W}^s$, $\nu$-almost every $x \in W$ has an unstable manifold of positive length.*

**Lemma 3.6.6.** *Let $\nu^{W_\xi}$ and $\hat{\nu}$ denote the conditional measures and factor measure obtained by disintegrating $\nu$ on the set of homogeneous stable manifolds $W_\xi \in \mathcal{W}^s_{\mathbb{H}}$, $\xi \in \Xi$. Then for any $\psi \in C^\alpha(M)$,*

$$\int_{W_\xi} \psi\, \mathrm{d}\nu^{W_\xi} = \frac{\int_{W_\xi} \psi \rho_\xi\, \nu}{\int_{W_\xi} \rho_\xi\, \nu} \quad \forall \xi \in \Xi, \ \text{and } \mathrm{d}\hat{\nu}(\xi) = |W_\xi|^{-1} \Big( \int_{W_\xi} \rho_\xi \nu \Big) \mathrm{d}\hat{\mu}_{SRB}(\xi).$$

*Moreover, viewed as a leafwise measure, $\nu(W) > 0$ for all $W \in \mathcal{W}^s$.*

*Proof of Lemma 3.6.6.* First, we establish the following claim: For $W \in \mathcal{W}^s$, we let $n_2 \leqslant \overline{C}_2|\log(|W|/\delta)|$ be the constant from the proof of Corollary 3.3.6 (This is the first time $l$ such that $\mathcal{G}_l^{\delta_1}(W)$ has at least one element of length at least $\delta_1/3$.) Then there exists $\overline{C} > 0$ such that for all $W \in \mathcal{W}^s$,

$$\int_W \nu \geqslant \overline{C}|W|^{(P_*(T,g)-\sup g)\overline{C}_2}. \tag{3.6.10}$$

Indeed, recalling (3.6.2) and using Theorem 3.5.3, we have for $\overline{C} = \frac{c_0}{2C}\delta_1^{1-(P_*(T,g)-\inf g)\overline{C}_2}$,

$$\begin{aligned}
\int_W \nu &= \lim_{n_j} \frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kP_*(T,g)} \int_W \mathcal{L}_g^k 1 \, \mathrm{d}m_W \\
&\geqslant \lim_{n_j} \frac{1}{n_j}\sum_{k=n_2}^{n_j-1} e^{-kP_*(T,g)} \sum_{W_i \in \mathcal{G}_{n_2}^{\delta_1}(W)} \int_{W_i} e^{S_{n_2}g}\mathcal{L}_g^{k-n_2} 1 \, \mathrm{d}m_{W_i} \\
&\geqslant \lim_{n_j} \frac{1}{n_j}\sum_{k=n_2}^{n_j-1} e^{-kP_*(T,g)} e^{n_2 \inf g} \frac{C\delta_1}{2} c_0 e^{P_*(T,g)(k-n_2)} \\
&\geqslant \frac{C\delta_1}{2} c_0 e^{-n_2(P_*(T,g)-\inf g)} \geqslant \overline{C}|W|^{(P_*(T,g)-\sup g)\overline{C}_2}.
\end{aligned}$$

This proves the last statement of the lemma.

Next, for any $f \in C^1(M)$, according to our convention, we view $f$ as an element of $\mathcal{B}_w$ by considering it as a measure integrated against $\mu_{\mathrm{SRB}}$. Now let $(\nu_{n_j})_j$ be the sequence of functions defined by (3.6.2) such that $|\nu_{n_j} - \nu|_w \to 0$. For any $\psi \in C^\alpha(M)$, we have

$$\begin{aligned}
\nu_{n_j}(\psi) &= \int_M \nu_{n_j}\psi \, \mathrm{d}\mu_{\mathrm{SRB}} = \int_\Xi \int_{W_\xi} \nu_{n_j}\psi\rho_\xi \, \mathrm{d}m_{W_\xi} |W_\xi|^{-1} \, \mathrm{d}\hat{\mu}_{\mathrm{SRB}}(\xi) \\
&= \int_\Xi \frac{\int_{W_\xi} \nu_{n_j}\psi\rho_\xi \, \mathrm{d}m_{W_\xi}}{\int_{W_\xi} \nu_{n_j}\rho_\xi \, \mathrm{d}m_{W_\xi}} \, \mathrm{d}(\hat{\mu}_{\mathrm{SRB}})_{n_j}(\xi)
\end{aligned}$$

where $\mathrm{d}(\hat{\mu}_{\mathrm{SRB}})_{n_j}(\xi) = |W_\xi|^{-1}\int_{W_\xi} \nu_{n_j}\rho_\xi \, \mathrm{d}m_{W_\xi} \, \mathrm{d}\hat{\mu}_{\mathrm{SRB}}(\xi)$. By definition of convergences in $\mathcal{B}_w$ since $\psi$, $\rho_\xi \in C^\alpha(W_\xi)$, the ratio of integrals converges (uniformly in $\xi$) to $\int_{W_\xi} \psi\rho_\xi\nu / \int_{W_\xi} \rho_\xi\nu$, and the factor measure converges to $|W_\xi|^{-1}(\int_{W_\xi} \rho_\xi \nu) \, \mathrm{d}\hat{\mu}_{\mathrm{SRB}}(\xi)$. Note that since $\rho_\xi$ is uniformly log-Hölder, and due to (3.6.10), we have $\int_{W_\xi} \rho_\xi\nu > 0$ with lower bound depending only on the length of $W_\xi$. Finally, by [BD20, Proposition 4.2] and [BD20, Lemma 4.4], we have $\nu_{n_j}(\psi)$ converging to $\nu(\psi)$. Disintegrating $\nu$ according the statement of the lemma yields to the claimed identifications. □

*Proof of Lemma 3.6.5.* The statement about stable manifolds of positive length follows from the characterization of $\hat{\nu}$ in Lemma 3.6.6, since the set of points with stable manifolds of zero length has zero $\hat{\mu}_{\mathrm{SRB}}$-measure [CM06].

We fix $W \in \mathcal{W}^s$ and prove the statement about $\nu$ as a leafwise measure. This will imply the statement regarding unstable manifolds for the measure $\nu$ by Lemma 3.6.6.

Fix $\varepsilon > 0$ and $\hat{\Lambda} \in (1, \Lambda)$, and define $O = \cup_{n\geqslant 1} O_n$, where

$$O_n := \{x \in W \mid n = \min\{j \geqslant 1 \mid d_u(T^{-j}x, \mathcal{S}_1) < \varepsilon C_e \hat{\Lambda}^{-j}\}\},$$

and $d_u$ denotes distance restricted to the unstable cone. By [CM06, Lemma 4.67], any $x \in W \setminus O$ has an unstable manifold of length at least $2\varepsilon$. We now estimate $\nu(O) = \sum_{n\geqslant 1} \nu(O_n)$, where equality holds since the $O_n$ are disjoint. Since each $O_n$ is a finite union of open subcurves of $W$, we have

$$\int_W \mathbb{1}_{O_n}\nu = \lim_{j\to\infty}\int_W \mathbb{1}_{O_n}\nu_{n_j} = \lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kP_*(T,g)}\int_W \mathbb{1}_{O_n}\mathcal{L}_g^k 1\, dm_W. \tag{3.6.11}$$

We give estimates in two cases.

*Case I: $k < n$.* Write $\int_{W\cap O_n}\mathcal{L}_g^k 1\, dm_W = \sum_{W_i\in\mathcal{G}_k^{\delta_0}(W)}\int_{W_i\cap T^{-k}O_n} e^{S_k g}\, dm_{W_i}$.

If $x \in T^{-k}O_n$, then $y = T^{-n+k}x$ satisfies $d_u(y, \mathcal{S}_1) < \varepsilon C_e \hat{\Lambda}^{-n}$ and thus we have $d_u(Ty, \mathcal{S}_{-1}) \leqslant C\varepsilon^{1/2}\hat{\Lambda}^{-n/2}$. Due to the uniform transversality of stable and unstable cones, as well as the fact that elements of $\mathcal{S}_{-1}$ are uniformly transverse to the stable cone, we have $d_s(Ty, \mathcal{S}_{-1}) \leqslant C\varepsilon^{1/2}\hat{\Lambda}^{-n/2}$ as well, with possibly a larger constant $C$.

Let $r^s_{-j}(x)$ denote the distance from $T^{-j}x$ to the nearest endpoint of $W^s(T^{-j}x)$, where $W^s(T^{-j}x)$ is the maximal local stable manifold containing $T^{-j}x$. From the above analysis, we see that $W_i \cap T^{-k}O_n \subseteq \{x \in W_i : r^s_{-n+k+1}(x) \leqslant C\varepsilon^{1/2}\hat{\Lambda}^{-n/2}\}$. The time reversal of the growth lemma [CM06, Thm 5.52] gives $m_{W_i}(r^s_{-n+k+1}(x) \leqslant C\varepsilon^{1/2}\hat{\Lambda}^{-n/2}) \leqslant C'\varepsilon^{1/2}\hat{\Lambda}^{-n/2}$ for a constant $C'$ that is uniform in $n$ and $k$. Thus, using Proposition 3.3.10, we find

$$\int_{W\cap O_n}\mathcal{L}_g^k 1\, dm_W \leqslant C'\varepsilon^{1/2}\hat{\Lambda}^{-n/2}\sum_{W_i\in\mathcal{G}_k^{\delta_0}(W)}|e^{S_k g}|_{C^0(W_i)} \leqslant Ce^{kP_*(T,g)}\varepsilon^{1/2}\hat{\Lambda}^{-n/2}\,.$$

*Case II: $k \geqslant n$.* Using the same observation as in Case I, if $x \in T^{-n+1}O_n$, then $x$ satisfies $d_s(x, \mathcal{S}_{-1}) \leqslant C\varepsilon^{1/2}\hat{\Lambda}^{-n/2}$. We change variables to estimate the integral precisely at time $-n+1$, and then use Propositions 3.5.1 and 3.3.10, and Lemma 3.3.12,

$$\begin{aligned}
\int_{W\cap O_n}\mathcal{L}^k 1\, dm_W &= \sum_{W_i\in\mathcal{G}_{n-1}^{\delta_0}(W)}\int_{W_i\cap T^{-n+1}O_n} e^{S_{n-1}g}\mathcal{L}_g^{k-n+1}1\, dm_{W_i}\\
&\leqslant \sum_{W_i\in\mathcal{G}_{n-1}^{\delta_0}(W)}\int_{W_i\cap(r_1^s\leqslant C\varepsilon^{1/2}\hat{\Lambda}^{-n/2})} e^{S_{n-1}g}\mathcal{L}_g^{k-n+1}1\, dm_{W_i}\\
&\leqslant \sum_{W_i\in\mathcal{G}_{n-1}^{\delta_0}(W)} |\log|W_i\cap(r_1^s\leqslant C\varepsilon^{1/2}\hat{\Lambda}^{-n/2})||^{-\gamma}|e^{S_{n-1}g}|_{C^\beta(W_i)}\|\mathcal{L}_g^{k-n+1}1\|_s\\
&\leqslant \sum_{W_i\in\mathcal{G}_{n-1}^{\delta_0}(W)} |\log(C\varepsilon^{1/2}\hat{\Lambda}^{-n/2})|^{-\gamma}C|e^{S_{n-1}g}|_{C^0(W_i)}e^{(k-n+1)P_*(T,g)}\\
&\leqslant |\log(C\varepsilon^{1/2}\hat{\Lambda}^{-n/2})|^{-\gamma}Ce^{kP_*(T,g)}\,.
\end{aligned}$$

Using the estimates of Cases I and II in (3.6.11) and using the weaker bound, we see that,

$$\int_W \mathbb{1}_{O_n}\nu_{n_j} \leqslant C|\log(C\varepsilon^{1/2}\hat{\Lambda}^{-n/2})|^{-\gamma}\,.$$

Summing over $n$, we have, $\int_W 1_O\,\nu_{n_j} \leqslant C'|\log\varepsilon|^{1-\gamma}$, uniformly in $j$. Since $\nu_{n_j}$ converges to $\nu$ in the weak norm, this bound carries over to $\nu$. Since $\varepsilon > 0$ was arbitrary and $\gamma > 1$, this implies $\nu(O) = 0$, completing the proof of the lemma. □

### 3.6.3 Absolute Continuity of $\mu_g$ – Full Support

In this subsection, we will assume that $\gamma > 1$, which is possible since $P_*(T,g) - \sup g > s_0 \log 2$. In the next subsection, we prove that $\mu_g$ is Bernoulli. This proof relies on showing first that $\mu_g$ is K-mixing (Proposition 3.6.12). As a first step, we will prove that $\mu_g$ is ergodic, using a Hopf-type argument. This will require the absolute continuity of the stable and the unstable foliations for $\mu_g$ (Corollary 3.6.8), which will be deduce from SSP.2 and the following absolute continuity for $\nu$:

**Proposition 3.6.7.** *Let $R$ be a Cantor rectangle. Fix $W^0 \in \mathcal{W}^s(R)$ and for $W \in \mathcal{W}^s(R)$, let $\Theta_W$ denote the holonomy map from $W^0 \cap R$ to $W \cap R$ along unstable manifolds in $\mathcal{W}^u(R)$. Then for any $(\mathcal{M}_0^1, \alpha_g)$-Hölder potential with $P_*(T,g) - \sup g > s_0 \log 2$ and having SSP.1, $\Theta_W$ is absolutely continuous with respect to the leafwise measure $\nu$.*

*Proof.* Since by Lemma 3.6.5 unstable manifolds comprise a set of full $\nu$-measure, it suffices to fix a set $E \subset W^0 \cap R$ with $\nu$-measure zero, and prove that the $\nu$-measure of $\Theta_W(E) \subset W$ is also zero.

Since $\nu$ is a regular measure on $W^0$, for $\varepsilon > 0$, there exists an open set $O_\varepsilon \subset W^0$, $O_\varepsilon \supset E$, such that $\nu(O_\varepsilon) \leqslant \varepsilon$. Indeed, since $W^0$ is compact, we may choose $O_\varepsilon$ to be a finite union of intervals. Let $\psi_\varepsilon$ be a smooth function which is 1 on $O_\varepsilon$ and 0 outside of an $\varepsilon$-neighbourhood of $O_\varepsilon$. We may choose $\psi_\varepsilon$ so that $\int_{W^0} \psi_\varepsilon \, \nu < 2\varepsilon$.

Using (3.5.4), we choose $n = n(\varepsilon)$ such that $|\psi_\varepsilon \circ T^n|_{C^1(T^{-n}W^0)} \leqslant 1$ and $\Lambda^{-n} \leqslant \varepsilon$. Following the procedure described in the proof of the estimate on the unstable norm in Proposition 3.5.1, we subdivide $T^{-n}W^0$ and $T^{-n}W$ into matched pieces $U_j^0$, $U_j$ and unmatched pieces $V_i^0$, $V_i$. With this construction, none of the unmatched pieces $T^nV_i^0$ intersect an unstable manifold in $\mathcal{W}^u(R)$ since unstable manifolds are not cut under $T^{-n}$.

Indeed, on matched pieces, we may choose a foliation $\Gamma_j = \{\gamma_x\}_{x \in U_j^0}$ such that:

i) $T^n\Gamma_j$ contains all unstable manifolds in $\mathcal{W}^u(R)$ that intersect $T^nU_j^0$;

ii) between unstable manifolds in $\Gamma_j \cap T^{-n}(\mathcal{W}^u(R))$, we interpolate via unstable curves;

iii) the resulting holonomy $\Theta_j$ from $T^nU_j^0$ to $T^nU_j$ has uniformly bounded Jacobian [16] with respect to arc-length, with bound depending on the unstable diameter of $D(R)$, by [BDL18, Lemmas 6.6, 6.8];

iv) pushing forward $\Gamma_j$ to $T^n\Gamma_j$ in $D(R)$, we interpolate in the gaps using unstable curves; call $\overline{\Gamma}$ the resulting foliation of $D(R)$;

v) the associated holonomy map $\overline{\Theta}_W$ extends $\Theta_W$ and has uniformly bounded Jacobian, again by [BDL18, Lemmas 6.6 and 6.8].

Using the map $\overline{\Theta}_W$, we define $\widetilde{\psi}_\varepsilon = \psi_\varepsilon \circ \overline{\Theta}_W^{-1}$, and note that $|\widetilde{\psi}_\varepsilon|_{C^1(W)} \leqslant C|\psi_\varepsilon|_{C^1(W^0)}$, where we write $C^1(W)$ for the set of Lipschitz functions on $W$, i.e., $C^\alpha$ with $\alpha = 1$.

Next, we modify $\psi_\varepsilon$ and $\widetilde{\psi}_\varepsilon$ as follows: We set them equal to 0 on the images of unmatched pieces, $T^nV_i^0$ and $T^nV_i$, respectively. Since these curves do not intersect unstable manifolds in $\mathcal{W}^u(R)$, we still have $\psi_\varepsilon = 1$ on $E$ and $\widetilde{\psi}_\varepsilon = 1$ on $\Theta_W(E)$. Moreover, the set of points on which $\psi_\varepsilon > 0$ (resp. $\widetilde{\psi}_\varepsilon > 0$) is a finite union of open intervals that cover $E$ (resp. $\Theta_W(E)$).

Since $\int_{W^0} \psi_\varepsilon \, \nu < 2\varepsilon$, in order to estimate $\int_W \widetilde{\psi}_\varepsilon \, \nu$, we estimate the following difference,

16. Indeed, [BDL18] shows the Jacobian is Hölder continuous, but we shall not need this here.

using matched pieces

$$\int_{W^0} \psi_\varepsilon \, \nu - \int_W \widetilde{\psi}_\varepsilon \, \nu = e^{-nP_*(T,g)} \Big( \int_{W^0} \psi_\varepsilon \, \mathcal{L}^n \nu - \int_W \widetilde{\psi}_\varepsilon \, \mathcal{L}^n \nu \Big)$$
$$= e^{-nP_*(T,g)} \sum_j \int_{U_j^0} \psi_\varepsilon \circ T^n \, e^{S_n g} \, \nu - \int_{U_j} \phi_j \, \nu + \int_{U_j} (\phi_j - \widetilde{\psi}_\varepsilon \circ T^n \, e^{S_n g}) \, \nu \,, \tag{3.6.12}$$

where $\phi_j = (\psi_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j^0} \circ G_{U_j}^{-1}$, and $G_{U_j^0}$ and $G_{U_j}$ represent the functions defining $U_j^0$ and $U_j$, respectively, defined as in (3.5.5). Next, since $d(\psi_\varepsilon \circ T^n \, e^{S_n g}, \phi_j) = 0$ by construction, and using (3.5.8) and the assumption that $\Lambda^{-n} \leqslant \varepsilon$, we have by (3.5.9),

$$e^{-nP_*(T,g)} \left| \sum_j \int_{U_j^0} \psi_\varepsilon \circ T^n \, \nu - \int_{U_j} \phi_j \, \nu \right| \leqslant C |\log \varepsilon|^{-\varsigma} \|\nu\|_u \,. \tag{3.6.13}$$

It remains to estimate the last term in (3.6.12). This we do using the weak norm,

$$\int_{U_j} (\phi_j - \widetilde{\psi}_\varepsilon \circ T^n \, e^{S_n g}) \, \nu \leqslant |\phi_j - \widetilde{\psi}_\varepsilon \circ T^n \, e^{S_n g}|_{C^\alpha(U_j)} \, |\nu|_w \,. \tag{3.6.14}$$

By (3.5.10), we have

$$|\phi_j - \widetilde{\psi}_\varepsilon \circ T^n \, e^{S_n g}|_{C^\alpha(U_j)} \leqslant C |(\psi_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j^0} - (\widetilde{\psi}_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j}|_{C^\alpha(I_j)} \,,$$

where $I_j$ is the common $r$-interval on which $G_{U_j^0}$ an $G_{U_j}$ are defined.

Fix $r \in I_j$, and let $x = G_{U_j^0}(r) \in U_j$ and $\bar{x} = G_{U_j}(r)$. Since $U_j^0$ and $U_j$ are matched, there exist $y \in U_j^0$ and an unstable curve $\gamma_y \in \Gamma_j$ such that $\gamma_y \cap U_j = \bar{x}$. By definition of $\widetilde{\psi}_\varepsilon$, we have $\widetilde{\psi}_\varepsilon \circ T^n(\bar{x}) = \psi_\varepsilon \circ T^n(y)$. Thus,

$$\begin{aligned}
&|(\psi_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j^0}(r) - (\widetilde{\psi}_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j}(r)| \\
&\quad \leqslant |\psi_\varepsilon \circ T^n(x) - \widetilde{\psi}_\varepsilon \circ T^n(\bar{x})||e^{S_n g(x)}| + |\widetilde{\psi}_\varepsilon \circ T^n(\bar{x})||e^{S_n g(x)} - e^{S_n g(\bar{x})}| \\
&\quad \leqslant (|\psi_\varepsilon \circ T^n(x) - \psi_\varepsilon \circ T^n(y)| + |\psi_\varepsilon \circ T^n(y) - \widetilde{\psi}_\varepsilon \circ T^n(\bar{x})|) e^{n \sup g} + |e^{S_n g(x)} - e^{S_n g(\bar{x})}| \\
&\quad \leqslant \Big( |\psi_\varepsilon \circ T^n|_{C^1(U_j^0)} d(x,y) + |g|_{C^{\alpha_g}} \frac{\Lambda^{\alpha_g}}{\Lambda^{\alpha_g} - 1} (C\varepsilon)^{\alpha_g} \Big) e^{n \sup g} \\
&\quad \leqslant (C\Lambda^{-n} + C\varepsilon^{\alpha_g}) e^{n \sup g} \leqslant C(\varepsilon + \varepsilon^{\alpha_g}) e^{n \sup g} \,,
\end{aligned}$$

where we have used the fact that $d(x,y) \leq C\Lambda^{-n}$ due to the uniform transversality of stable and unstable curves. We also used the fact that, by definition, the vertical segment $\gamma_x$ connecting $x$ to $\bar{x}$ is such that $|T^n \gamma_x| < C\varepsilon$. Since each $T^i \gamma_x$ lies in the extended unstable cone, for all $0 \leqslant i \leqslant n$, we get that $d(T^i(x), T^i(\bar{x})) \leqslant C\Lambda^{-(n-i)}\varepsilon$, hence the bound

$$\begin{aligned}
|e^{S_n g(x)} - e^{S_n g(\bar{x})}| &\leqslant |e^{S_n g(x)}| \cdot |1 - e^{S_n g(\bar{x}) - S_n g(x)}| \leqslant 2 e^{n \sup g} |S_n g(\bar{x}) - S_n g(x)| \\
&\leqslant \frac{\Lambda^{\alpha_g}}{\Lambda^{\alpha_g} - 1} (C\varepsilon)^{\bar{\alpha}} |g|_{C^{\alpha_g}} e^{n \sup g}
\end{aligned}$$

where we used that $|1 - e^x| \leqslant 2|x|$ when $x$ is near 0.

Now given $r, s \in I_j$, we have on the one hand,

$$\begin{aligned}
&\big|(\psi_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j^0}(r) - (\widetilde{\psi}_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j}(r) \\
&\quad - (\psi_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j^0}(s) + (\widetilde{\psi}_\varepsilon \circ T^n \, e^{S_n g}) \circ G_{U_j}(s)\big| \leqslant 2C\varepsilon^{\bar{\alpha}} e^{n \sup g} \,,
\end{aligned} \tag{3.6.15}$$

while on the other hand,

$$\begin{aligned}
&|(\psi_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j^0}(r) - (\widetilde{\psi}_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j}(r) \\
&\qquad - (\psi_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j^0}(s) + (\widetilde{\psi}_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j}(s)| \\
&= |(\psi_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j^0}(r) - (\psi_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j^0}(s) \\
&\qquad - ((\widetilde{\psi}_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j}(r) - (\widetilde{\psi}_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j}(s))| \\
&\leqslant |\psi_\varepsilon|_{C^0(W^0)} |e^{S_n g}|_{C^{\alpha_g}} d(G_{U_j^0}(r), G_{U_j^0}(s))^{\alpha_g} + |\psi_\varepsilon \circ T^n|_{C^1(W^0)} d(G_{U_j^0}(r), G_{U_j^0}(s)) |e^{S_n g}|_{C^0} \\
&\qquad + |\widetilde{\psi}_\varepsilon|_{C^0(W)} |e^{S_n g}|_{C^{\alpha_g}} d(G_{U_j^0}(r), G_{U_j^0}(s))^{\alpha_g} + |\widetilde{\psi}_\varepsilon \circ T^n|_{C^1(W)} d(G_{U_j^0}(r), G_{U_j^0}(s)) |e^{S_n g}|_{C^0} \\
&\leqslant (C|r-s| + C'|r-s|^{\alpha_g}) e^{n \sup g} \leqslant C|r-s|^{\alpha_g} e^{n \sup g}\,,
\end{aligned} \tag{3.6.16}$$

where we have used Lemma 3.3.12 and the fact that $G^{-1}_{U_j^0}$ and $G^{-1}_{U_j}$ have bounded derivatives since the stable cone is bounded away from the vertical.

The difference between evaluations at $r$ and $s$, divided by $|r-s|^\alpha$ is bounded by the minimum of expressions (3.6.15) and (3.6.16), both divided by $|r-s|^\alpha$. This is greatest when the two are equal, i.e., when $|r-s| = C\varepsilon$. Thus $H^\alpha((\psi_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j^0} - (\widetilde{\psi}_\varepsilon \circ T^n e^{S_n g}) \circ G_{U_j}) \leqslant C\varepsilon^{\alpha_g - \alpha} e^{n \sup g}$, and so $|\phi_j - \widetilde{\psi}_\varepsilon \circ T^n e^{S_n g}|_{C^\alpha(U_j)} \leqslant C\varepsilon^{\alpha_g - \alpha} e^{n \sup g}$. Putting this estimate together with (3.6.13) and (3.6.14) in (3.6.12), we conclude,

$$\left| \int_{W^0} \psi_\varepsilon\, \nu - \int_W \widetilde{\psi}_\varepsilon\, \nu \right| \leqslant C|\log \varepsilon|^{-\varsigma} \|\nu\|_u + C\varepsilon^{\alpha_g - \alpha} |\nu|_w e^{-n(P_*(T,g) - \sup g)}\,. \tag{3.6.17}$$

Now since $\int_{W^0} \psi_\varepsilon\, \nu \leqslant 2\varepsilon$, we have

$$\int_W \widetilde{\psi}_\varepsilon\, \nu \leqslant C' |\log \varepsilon|^{-\varsigma}\,, \tag{3.6.18}$$

where $C'$ depends on $\nu$. Since $\widetilde{\psi}_\varepsilon = 1$ on $\Theta_W(E)$ and $\widetilde{\psi}_\varepsilon > 0$ on an open set containing $\Theta_W(E)$ for every $\varepsilon > 0$, we have $\nu(\Theta_W(E)) = 0$, as required. □

**Corollary 3.6.8** (Absolute Continuity of $\mu_g$ with Respect to Unstable Foliations)**.** *Let $R$ be a Cantor rectangle with $\mu_g(R) > 0$. Fix $W^0 \in \mathcal{W}^s(R)$ and for $W \in \mathcal{W}^s(R)$, let $\Theta_W$ denote the holonomy map from $W^0 \cap R$ to $W \cap R$ along unstable manifolds in $\mathcal{W}^u(R)$. Then $\Theta_W$ is absolutely continuous with respect to the measure $\mu_g$.*

In order to deduce the corollary from the Proposition 3.6.7, we introduce the set $M^{\rm reg}$, as in [BD20], of regular points and a countable cover of this set by Cantor rectangles. The set $M^{\rm reg}$ is defined by

$$M^{\rm reg} = \{x \in M \mid d(x, \partial W^s(x)) > 0\,,\ d(x, \partial W^u(x)) > 0\}.$$

At each $x \in M^{\rm reg}$, we can apply [CM06, Prop 7.81] and construct a closed locally maximal Cantor rectangle $R_x$ containing $x$, which is the direct product of local stable and unstable manifolds. Furthermore, by trimming the sides, we may arrange it so that $\frac{1}{2}{\rm diam}^s(R_x) \leqslant {\rm diam}^u(R_x) \leqslant 2\,{\rm diam}^s(R_x)$.

**Lemma 3.6.9** (Countable Cover of $M^{\rm reg}$ by Cantor Rectangle)**.** *There exists a countable set $\{x_j\}_{j \in \mathbb{N}} \subset M^{\rm reg}$ such that $\cup_j R_{x_j} = M^{\rm reg}$ and each $R_j := R_{x_j}$ satisfies (3.3.18).*

*Proof.* This lemma is exactly the content of [BD20, Lemma 7.10]. □

Let $\{R_j \mid j \in \mathbb{N}\}$ be the family of Cantor rectangles constructed in Lemma 3.6.9, discarding the ones with zero $\mu_g$-measure. Then $\mu_g(\cup_j R_j) = \mu_g(M^{\mathrm{reg}}) = 1$, by Corollary 3.6.3(d). In the rest of the paper, we shall work with this countable collection of rectangles.

Given a Cantor rectangle $R$, define $\mathcal{W}^s(R)$ to be the set of stable manifolds that completely cross $D(R)$, and similarly for $\mathcal{W}^u(R)$.

*Proof of Corollary 3.6.8.* In order to prove absolute continuity of the unstable foliation with respect to $\mu_g$, we will show that the conditional measures $\mu_g^W$ of $\mu_g$ are equivalent to $\nu$ on $\mu_g$-almost every $W \in \mathcal{W}^s(R)$.

Fix a Cantor rectangle $R$ satisfying (3.3.18) with $\mu_g(R) > 0$, and $W^0$ as in the statement of Corollary 3.6.8. Let $E \subset W^0 \cap R$ satisfy $\nu(E) = 0$, for the leafwise measure $\nu$.

For any $W \in \mathcal{W}^s(R)$, we have the holonomy map $\Theta_W : W^0 \cap R \to W \cap R$ as in the proof of Proposition 3.6.7. For $\varepsilon > 0$, we approximate $E$, choose $n$ and construct a foliation $\overline{\Gamma}$ of the solid rectangle $D(R)$ as before. Define $\psi_\varepsilon$ and use the foliation $\overline{\Gamma}$ to define $\widetilde{\psi}_\varepsilon$ on $D(R)$. We have $\widetilde{\psi}_\varepsilon = 1$ on $\bar{E} = \cup_{x\in E}\bar{\gamma}_x$, where $\bar{\gamma}_x$ is the element of $\overline{\Gamma}$ containing $x$. We extend $\widetilde{\psi}_\varepsilon$ to $M$ by setting it equal to 0 on $M \setminus D(R)$.

It follows from the proof of Proposition 3.6.7, in particular (3.6.18), that $\widetilde{\psi}_\varepsilon \nu \in \mathcal{B}_w$, and $|\widetilde{\psi}_\varepsilon \nu|_w \leqslant C'|\log\varepsilon|^{-\varsigma}$. Now,

$$\begin{aligned}\tilde{\nu}(\nu)\,\mu_g(\widetilde{\psi}_\varepsilon) = \tilde{\nu}(\widetilde{\psi}_\varepsilon\nu) &= \lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kP_*(T,g)}(\mathcal{L}_g^*)^k\,\mathrm{d}\mu_{\mathrm{SRB}}(\widetilde{\psi}_\varepsilon\nu)\\ &= \lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1} e^{-kP_*(T,g)}\mu_{\mathrm{SRB}}(\mathcal{L}_g^k(\widetilde{\psi}_\varepsilon\nu))\,.\end{aligned} \tag{3.6.19}$$

For each $k$, using the disintegration of $\mu_{\mathrm{SRB}}$ as in the proof of Lemma 3.6.6 with the same notation as there, and (3.6.5), we estimate,

$$\begin{aligned}\mu_{\mathrm{SRB}}(\mathcal{L}_g^k(\widetilde{\psi}_\varepsilon\nu)) &= \int_\Xi\int_{W_\xi}\mathcal{L}_g^k(\widetilde{\psi}_\varepsilon\nu)\,\rho_\xi\,\mathrm{d}m_{W_\xi}\,|W_\xi|^{-1}\,\mathrm{d}\hat{\mu}_{\mathrm{SRB}}(\xi)\\ &\leqslant C_\rho\int_\Xi|\mathcal{L}_g^k(\widetilde{\psi}_\varepsilon\nu)|_w\,|W_\xi|^{-1}\,\mathrm{d}\hat{\mu}_{\mathrm{SRB}}(\xi)\\ &\leqslant C_\rho\, e^{kP_*(T,g)}|\widetilde{\psi}_\varepsilon\nu|_w \leqslant Ce^{kP_*(T,g)}|\log\varepsilon|^{-\varsigma}\,,\end{aligned}$$

where we have used (3.5.1) in the last line, as well as the $\hat{\mu}_{\mathrm{SRB}}$-integrability of $|W_\xi|^{-1}$ from [CM06, Exercise 7.22]. Thus $\mu_g(\widetilde{\psi}_\varepsilon) \leqslant C|\log\varepsilon|^{-\varsigma}$, for each $\varepsilon > 0$, so that $\mu_g(\bar{E}) = 0$.

Disintegrating $\mu_g$ into conditional measures $\mu_g^{W_\xi}$ on $W_\xi \in \mathcal{W}^s$ and a factor measure $d\hat{\mu}_g(\xi)$ on the index set $\Xi_R$ of stable manifolds in $\mathcal{W}^s(R)$, it follows that $\mu_g^{W_\xi}(\bar{E}) = 0$ for $\hat{\mu}_g$-almost every $\xi \in \Xi_R$. Since $E$ was arbitrary, the conditional measures of $\mu_g$ on $\mathcal{W}^s(R)$ are absolutely continuous with respect to the leafwise measure $\nu$.

To show that in fact $\mu_g^W$ is equivalent to $\nu$, suppose now that $E \subset W^0$ has $\nu(E) > 0$. For any $\varepsilon > 0$ such that $C'|\log\varepsilon|^{-\varsigma} < \nu(E)/2$, where $C'$ is from (3.6.18), choose $\psi_\varepsilon \in C^1(W^0)$ such that $\nu(|\psi_\varepsilon - 1_E|) < \varepsilon$, where $1_E$ is the indicator function of the set $E$. As above, we extend $\psi_\varepsilon$ to a function $\widetilde{\psi}_\varepsilon$ on $D(R)$ via the foliation $\overline{\Gamma}$, and then to $M$ by setting $\widetilde{\psi}_\varepsilon = 0$ on $M \setminus D(R)$.

We have $\widetilde{\psi}_\varepsilon \nu \in \mathcal{B}_w$ and by (3.6.17)

$$\nu(\widetilde{\psi}_\varepsilon 1_W) \geqslant \nu(\psi_\varepsilon 1_{W^0}) - C'|\log \varepsilon|^{-\varsigma}, \qquad \text{for all } W \in \mathcal{W}^s(R). \tag{3.6.20}$$

Now following (3.6.19) and disintegrating $\mu_{\mathrm{SRB}}$ as usual, we obtain,

$$\begin{aligned} \mu_g(\widetilde{\psi}_\varepsilon) &= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} e^{-kP_*(T,g)} \int_\Xi \int_{W_\xi} \mathcal{L}_g^k(\widetilde{\psi}_\varepsilon \nu)\, \rho_\xi \, dm_{W_\xi} \, d\hat{\mu}_{\mathrm{SRB}}(\xi) \\ &= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} e^{-kP_*(T,g)} \int_\Xi \left( \sum_{W_{\xi,i} \in \mathcal{G}_k^{\delta_1}(W_\xi)} \int_{W_{\xi,i}} \widetilde{\psi}_\varepsilon \, \rho_\xi \circ T^k \, e^{S_k g} \, \nu \right) d\hat{\mu}_{\mathrm{SRB}}(\xi) \, . \end{aligned} \tag{3.6.21}$$

To estimate this last expression, we estimate the thermodynamic sum over the curves $W_{\xi,i}$ which properly cross the rectangle $R$.

By SSP.2 and the choice of $\delta_1$ in (3.3.8), there exists $k_0$, depending only on the minimum length of $W \in \mathcal{W}^s(R)$, such that

$$\sum_{W_i \in L_k^{\delta_1}(W_\xi)} |e^{S_k g}|_{C^0(W_i)} \geqslant \frac{1}{3} \sum_{W_i \in \mathcal{G}_k^{\delta_1}(W_\xi)} |e^{S_k g}|_{C^0(W_i)} \, , \quad \text{for all } k \geqslant k_0.$$

By choice of our covering $\{R_i\}$ from Lemma 3.6.9, all $W_{\xi,j} \in L_k^{\delta_1}(W_\xi)$ properly cross one of finitely many $R_i$. By the topological mixing property of $T$, there exists $n_0$, depending only on the length scale $\delta_1$, such that some smooth component of $T^{-n_0} W_{\xi,j}$ properly crosses $R$. Thus, letting $\mathcal{C}_k(W_\xi)$ denote those $W_{\xi,i} \in \mathcal{G}_k^{\delta_1}(W_\xi)$ which properly cross $R$, we have

$$\begin{aligned} \sum_{W_i \in \mathcal{C}_{k+n_0}(W_\xi)} |e^{S_k g}|_{C^0(W_i)} &\geqslant \sum_{W_{\xi,i} \in L_k^{\delta_1}(W_\xi)} \sum_{\tilde{W} \subset \mathcal{G}_{n_0}^{\delta_1}(W_{\xi,i}) \cap \mathcal{C}_{k+n_0}(W_\xi)} e^{n_0 \inf g} |e^{S_k g}|_{C^0(W_{\xi,i})} \\ &\geqslant e^{n_0 \inf g} \sum_{W_{\xi,i} \in L_k^{\delta_1}(W_\xi)} |e^{S_k g}|_{C^0(W_{\xi,i})} \\ &\geqslant \frac{1}{3} e^{n_0 \inf g} \sum_{W_{\xi,i} \in \mathcal{G}_k^{\delta_1}(W_\xi)} |e^{S_k g}|_{C^0(W_{\xi,i})} \geqslant \frac{1}{3} c \, e^{n_0 \inf g} \, e^{kP_*(T,g)} \, , \end{aligned}$$

for all $k \geqslant k_0$, where $c > 0$ depends on $c_0$ from Proposition 3.3.7 as well as the minimum length of $W \in \mathcal{W}^s(R)$.

Using this lower bound on the sum together with (3.6.20) yields,

$$\mu_g(\widetilde{\psi}_\varepsilon) \geqslant \tfrac{1}{3} c e^{-n_0 P_*(T,g)} \big(\nu(\psi_\varepsilon) - C'|\log \varepsilon|^{-\varsigma}\big) \geqslant C''\big(\nu(E) - |\log \varepsilon|^{-\varsigma}\big) \, .$$

Taking $\varepsilon \to 0$, we have

$$\mu_g(\bar{E}) \geqslant C'' \nu(E), \tag{3.6.22}$$

and so $\mu_g^W(\bar{E}) > 0$ for almost every $W \in \mathcal{W}^s(R)$. □

**Proposition 3.6.10** (Full Support). *We have $\mu_g(O) > 0$ for any open set $O$.*

*Proof.* The proof is the same as the one of [BD20, Proposition 7.11], replacing $\mu_*$ by $\mu_g$. □

### 3.6.4 Bernoulli property of $\mu_g$ and Variational Principle

In this section, we use the absolute continuity results on the holonomy map from Section 3.6.3 to establish that $\mu_g$ is K-mixing. We also prove an upper bound on the $\mu_g$-measure of weighted dynamical Bowen balls. Using these estimates, we are able to prove that $\mu_g$ is an equilibrium state for $T$ under the potential $g$ – that is, $\mu_g$ realizes the sup in the definition of $P(T, g)$ – and $\mu_g$ satisfies the variational principle: $P_*(T, g) = P(T, g)$. Finally, using again the absolute continuity along side with Cantor rectangles and the bound (3.6.6) on the neighbourhoods of the singular sets, we can bootstrap from the K-mixing to prove that $\mu_g$ is Bernoulli.

**Lemma 3.6.11** (Single Ergodic Component)**.** *If $R$ is a Cantor rectangle with $\mu_g(R) > 0$, then all the stable manifolds $\mathcal{W}^s(R)$ are contained in a single ergodic component of $\mu_g$.*

*Proof.* Replacing $\mu_*$ by $\mu_g$, the proof of the analogous result [BD20, Lemma 7.15] can be applied verbatim. The proof there follows the Hopf strategy. □

**Proposition 3.6.12.** *For all $(\mathcal{M}_0^1, \alpha_g)$-Hölder potential $g$ such that $P_*(T, g) - \sup g > s_0 \log 2$ and having SSP.1 and SSP.2, $(T, \mu_g)$ is K-mixing.*

*Proof.* Replacing $\mu_*$ by $\mu_g$, the proof of the analogous result [BD20, Proposition 7.16] can be applied verbatim. We outline the steps of the proof.

First, Baladi and Demers show that $(T^n, \mu_*)$ is ergodic for all $n \geqslant 1$. To do so, they use the topological mixing of $T$ to prove that any two Cantor rectangles belong to the same ergodic component of $T^n$.

Then, they prove that $T$ is K-mixing. To do so, they construct a measurable partition out of the stable and unstable manifolds, that is finer than the Pinsker partition $\pi(T)$. Using the covering of $M^{\text{reg}}$ by Cantor rectangles $\{R_i\}$, and the absolute continuity of the holonomy map, they prove that each $R_i$ belongs to a single component of $\pi(T)$. From this, they deduce that $\pi(T)$ contains finitely many elements on which $T$ acts by permutation. Since $\pi(T)$ is $T$-invariant and $(T^n, \mu_*)$ is ergodic for all $n \geqslant 1$, $\pi(T)$ must be trivial. □

**Proposition 3.6.13** (Upper Bounds on Weighted Dynamical Balls)**.** *Assume that $P_*(T, g) - \sup g > s_0 \log 2$ and that SSP.1 holds. There exists $A < \infty$ such that for all $\varepsilon > 0$ sufficiently small, $x \in M$, and $n \geqslant 1$, the measure $\mu_g$ constructed in (3.6.1) satisfies*

$$\mu_g(e^{-S_n^{-1} g} \mathbb{1}_{B_n^{-1}(x,\varepsilon)}) \leqslant A e^{-n P_*(T,g)} ,$$

*where $B_n^{-1}(x, \varepsilon)$ is the Bowen ball at $x$ of length $n$ for $T^{-1}$.*

*Proof.* The inequality follows from the beginning of the proof of [BD20, Proposition 7.12], where $\mu_*, \mathcal{L}$ and $h_*$ should be replaced by respectively $\mu_g, \mathcal{L}_g$ and $P_*(T, g)$. □

**Corollary 3.6.14.** *For all $(\mathcal{M}_0^1, \alpha_g)$-Hölder potential $g$ such that $P_*(T, g) - \sup g > s_0 \log 2$ and having SSP.1 and SSP.2, the measure $\mu_g$ is an equilibrium state of $T$ under the potential $g$: we have $P_*(T, g) = h_{\mu_g}(T) + \int g \, \mathrm{d}\mu_g$.*

*Proof.* For all $x \in M$, let $\mathcal{P}_{-n}^0(x)$ denotes the element of $\mathcal{P}_{-n}^0$ containing $x$. By the Shannon–MacMillan–Breiman theorem, we have

$$-\lim_{n\to\infty} \frac{1}{n} \log \mu_g(\mathcal{P}_{-n}^0(x)) = h_{\mu_g}(\mathcal{P}, T^{-1}) = h_{\mu_g}(T) \quad \text{for } \mu_g\text{-a.e. } x \in M ,$$

where the last equality follows from the Kolmogorov–Sinai theorem (because $T$ is expansive [BD20, Lemma 3.4]). Furthermore, since by Proposition 3.6.12 $\mu_g$ is ergodic, then $\frac{1}{n}\log e^{-S_n^{-1}g}$ converges to $-\mu_g(g)$ as $n$ goes to infinity. Thus

$$-\lim_{n\to\infty}\frac{1}{n}\log\Big(e^{-S_n^{-1}g(x)}\mu_g(\mathcal{P}^0_{-n}(x))\Big)=h_{\mu_g}(T)+\int g\,\mathrm{d}\mu_g \quad \text{for } \mu_g\text{-a.e. } x\in M\,. \tag{3.6.23}$$

Now, by Lemma 3.2.3, there exists a constant $C$ such that for all $x\in M$ and all $y,z\in\mathcal{P}^0_{-n}(x)$, we have $|S_n^{-1}g(y)-S_n^{-1}g(z)|\leqslant C$. Thus

$$e^{-C}\leqslant\frac{\mu_g\Big(e^{-S_n^{-1}g}\mathbb{1}_{\mathcal{P}^0_{-n}(x)}\Big)}{e^{-S_n^{-1}g(x)}\mu_g(\mathcal{P}^0_{-n}(x))}\leqslant e^{C}\,,$$

and so we can replace $e^{-S_n^{-1}g(x)}\mu_g(\mathcal{P}^0_{-n}(x))$ in (3.6.23) by $\mu_g\Big(e^{-S_n^{-1}g}\mathbb{1}_{\mathcal{P}^0_{-n}(x)}\Big)$.

Now, we want to replace $\mathcal{P}^0_{-n}(x)$ with a dynamical ball and use Proposition 3.6.13. To do so, recall that for all $\varepsilon<\varepsilon_0$, the dynamical ball $B_n(x,\varepsilon)$ is included in a single element of $\mathcal{M}_0^n$, which is itself included in at most $C$ elements of $\mathcal{P}_0^n$, for some $C$ independent of $x$. Thus, using time reversals

$$-\liminf_{n\to\infty}\frac{1}{n}\log\mu_g\Big(e^{-S_n^{-1}g}\mathbb{1}_{B_n^{-1}(x,\varepsilon)}\Big)\leqslant h_{\mu_g}(T)+\int g\,\mathrm{d}\mu_g,$$

On the other hand, for $\varepsilon$ small enough, we get by Proposition 3.6.13,

$$-\liminf_{n\to\infty}\frac{1}{n}\log\mu_g\Big(e^{-S_n^{-1}g}\mathbb{1}_{B_n^{-1}(x,\varepsilon)}\Big)\geqslant P_*(T,g).$$

Combining these last two inequalities, we get $h_{\mu_g}(T)+\int g\,\mathrm{d}\mu_g\geqslant P_*(T,g)$, which ends the proof. □

**Proposition 3.6.15.** *Under the assumptions of Proposition 3.6.12, $(T,\mu_g)$ is Bernoulli.*

*Proof.* The proof follows the arguments in Section 5 and 6 in [CH96], relying on the notion of vwB partitions introduced by Ornstein in [Orn70]. Actually, we can apply the same modifications as in the proof of the analogous result [BD20, Proposition 7.19], replacing $\mu_*$ by $\mu_g$. □

### 3.6.5 Uniqueness of the equilibrium state

This subsection is devoted to the uniqueness of the equilibrium state $\mu_g$ (Proposition 3.6.18). The proof relies on exploiting the fact that while the lower bound on weighted Bowen balls (or thermodynamic sum over elements of $\mathcal{M}^0_{-n}$) cannot be improved for $\mu_g$-almost every $x$, yet if one fixes $n$, most elements of $\mathcal{M}^0_{-n}$ (in the sense of thermodynamic sums) should either have unstable diameter of a fixed length, or have previously been contained in an element of $\mathcal{M}^0_{-j}$ with this property, for some $j<n$ (Lemma 3.6.16). Such elements collectively satisfy stronger lower bounds on their measure, when weighted accordingly (Lemma 3.6.17). Since we have establish good control of the sums over $\mathcal{M}^0_{-n}$ and $\mathcal{M}^n_0$ in Section 3.3, we will work with these partitions instead of Bowen balls.

Recalling (3.3.1), choose $m_1$ large enough so that $(Km_1+1)^{1/m_1}<e^{\frac{1}{4}(P_*(T,g)-\sup g)}$. Now, choose $\delta_2>0$ sufficiently small that for all $n$, $k\in\mathbb{N}$, if $A\in\mathcal{M}^k_{-n}$ is such that

$$\max\{\operatorname{diam}^u(A),\,\operatorname{diam}^s(A)\}\leqslant\delta_2,$$

then $A \smallsetminus \mathcal{S}_{\pm m_1}$ consists of at most $Km_1 + 1$ connected components.

For $n \geqslant 1$, define

$$B^0_{-2n} \coloneqq \{A \in \mathcal{M}^0_{-2n} \mid \forall\, 0 \leqslant j \leqslant n/2\,,\\ T^{-j}A \subset E \in \mathcal{M}^0_{-2n+j} \text{ such that } \operatorname{diam}^u(E) < \delta_2\},$$

and its time reversal

$$B_0^{2n} \coloneqq \{A \in \mathcal{M}_0^{2n} \mid \forall\, 0 \leqslant j \leqslant n/2\,,\\ T^{j}A \subset E \in \mathcal{M}_0^{2n-j} \text{ such that } \operatorname{diam}^s(E) < \delta_2\}.$$

Next, set $B_{2n} = \{A \in \mathcal{M}^0_{-2n} \mid \text{ either } A \in B^0_{-2n} \text{ or } T^{-2n}A \in B_0^{2n}\}$. Define $G_{2n} = \mathcal{M}^0_{-2n} \smallsetminus B_{2n}$.

Our first lemma shows that the thermodynamic sum over elements of $B_{2n}$ is small relative to the one over elements of $\mathcal{M}^0_{-2n}$, for large $n$. Let $n_1 \geqslant 2m_1$ be chosen so that for all $A \in \mathcal{M}^0_{-n}$, $\operatorname{diam}^s(A) \leqslant C\Lambda^{-n} \leqslant \delta_2$ for all $n \geqslant n_1$.

**Lemma 3.6.16.** *There exists $C > 0$ such that for all $n \geqslant n_1$,*

$$\sum_{A \in B_{2n}} |e^{S^{-1}_{2n} g}|_{C^0(A)} \leqslant C e^{\frac{3}{2}nP_*(T,g)} e^{\frac{1}{2}n \sup g} (Km_1+1)^{\frac{n}{m_1}+1} \leqslant C e^{\frac{7}{4}nP_*(T,g) + \frac{1}{4}n\sup g}\,.$$

Notice that since $P_*(T,g) - \sup g > 0$, we have that $\frac{7}{4}P_*(T,g) + \frac{1}{4}\sup g < 2P_*(T,g)$.

*Proof.* Let $n \geqslant n_1$ and $A \in B^0_{-2n} \subset \mathcal{M}^0_{-2n}$. For all $0 \leqslant j \leqslant \lfloor n/2 \rfloor$, call $A_j \in \mathcal{M}^0_{-\lceil 3n/2\rceil - j}$ the unique element containing $T^{-\lfloor n/2\rfloor + j}A$. By definition of $B^0_{-2n}$, we have that $\operatorname{diam}^u(A_j) \leqslant \delta_2$, meanwhile $\operatorname{diam}^s(A_j) \leqslant \delta_2$ by choice of $n_1$.

By choice of $\delta_2$, we have that $A_0$ is the union of at most $Km_1+1$ elements of $\mathcal{M}^{m_1}_{-\lceil 3n/2\rceil}$. Thus the number of connected components of $T^{m_1}A_0$ is at most $Km_1 + 1$. Notice that this fact not only applies to $A_0$, but also to $A_{m_1}, \dots, A_{lm_1}, A_{\lfloor n/2\rfloor}$, where $\lfloor n/2 \rfloor = lm_1 + i$, $0 \leqslant i < m_1$. Thus, we get

$$\#\{A' \in B^0_{-2n} \mid T^{-\lfloor n/2\rfloor}A' \subset A_0\} \leqslant (Km_1+1)^{l+1} \leqslant (Km_1+1)^{\frac{n}{m_1}+1}\,.$$

We are now able to estimate the thermodynamic sum over $B^0_{-2n}$:

$$\begin{aligned}
\sum_{A \in B^0_{-2n}} |e^{S^{-1}_{2n}g}|_{C^0(A)} &= \sum_{A_0 \in \mathcal{M}^0_{-\lceil 3n/2\rceil}} \sum_{\substack{A' \in B^0_{-2n} \\ T^{-\lfloor n/2\rfloor}A' \subset A_0}} |e^{S^{-1}_{2n}g}|_{C^0(A')} \\
&= \sum_{A_0}\sum_{A'} \left| e^{S^{-1}_{\lceil 3n/2\rceil} g\circ T^{\lfloor n/2\rfloor} + S^{-1}_{\lfloor n/2\rfloor} g} \right|_{C^0(A')} \leqslant \sum_{A_0} \left| e^{S^{-1}_{\lceil 3n/2\rceil} g}\right|_{C^0(A_0)} \sum_{A'} \left| e^{S^{-1}_{\lfloor n/2\rfloor} g}\right|_{C^0(A')} \\
&\leqslant e^{\frac{1}{2}n\sup g}(Km_1+1)^{\frac{n}{m_1}+1} \sum_{A_0} \left| e^{S^{-1}_{\lceil 3n/2\rceil} g}\right|_{C^0(A_0)} \leqslant C e^{\frac{7}{4}nP_*(T,g) + \frac{1}{4}n\sup g}\,,
\end{aligned}$$

where we used Proposition 3.3.10 for the last inequality.

Now, notice that $B_0^{2n}$ is the time reversal of $B^0_{-2n}$, thus

$$\sum_{A \in B_0^{2n}} |e^{S_{2n}g}|_{C^0(A)} \leqslant C e^{\frac{7}{4}nP_*(T^{-1},g) + \frac{1}{2}n\sup g} = C e^{\frac{7}{4}nP_*(T,g) + \frac{1}{4}n\sup g}.$$

Hence

$$\sum_{A\in B_0^{2n}} |e^{S_{2n}^{-1}g}|_{C^0(T^{-2n}A)} = \sum_{A\in B_0^{2n}} |e^{S_{2n}g}|_{C^0(A)} \leqslant Ce^{\frac{7}{4}nP_*(T,g)+\frac{1}{4}n\sup g}.$$

Finally, we get

$$\sum_{A\in B_{2n}} |e^{S_{2n}^{-1}g}|_{C^0(A)} \leqslant 2Ce^{\frac{7}{4}nP_*(T,g)+\frac{1}{4}n\sup g}.$$

□

Next, the following lemma establishes the importance of long pieces in providing good lower bounds on the measure of weighted elements of the partition.

**Lemma 3.6.17.** *There exists $C_{\delta_2} > 0$ such that for all $j \geqslant 1$ and all $A \in \mathcal{M}_{-j}^0$ such that diam$^u(A) \geqslant \delta_2$ and diam$^s(T^{-j}A) \geqslant \delta_2$, we have*

$$\mu_g(e^{-S_j^{-1}g}\mathbb{1}_A) \geqslant C_{\delta_2}e^{-jP_*(T,g)}\,.$$

*Proof.* Let $R_1, \ldots, R_k$ be Cantor rectangles such that $\mu_g(R_i) > 0$ for all $1 \leqslant i \leqslant k$, and such that any unstable or stable curve of length more than $\delta_2$ fully crosses at least one of them. Note $\mathcal{R}_{\delta_2} = \{R_1, \ldots, R_k\}$ this family.

Let $j > 0$ and $A \in \mathcal{M}_{-j}^0$ such that diam$^u(A) \geqslant \delta_2$ and diam$^s(T^{-j}A) \geqslant \delta_2$. By choice of $\mathcal{R}_{\delta_2}$, $A$ crosses some rectangle $R_i$ and $T^{-j}A$ also crosses some rectangle $R_{i'}$. Note $\Xi_i$ the index set for the family of stable manifolds $W_\xi$ of $R_i$. For $\xi \in \Xi_i$, let $W_{\xi,A} := W_\xi \cap A$. Since $T^{-j}A$ properly crosses $R_{i'}$ in the stable direction, and that $T^{-j}$ is smooth on $A$, it follows that $T^{-j}W_{\xi,A}$ is a single curve containing a stable manifold of $R_{i'}$.

Let $l_{\delta_2}$ denote the length of the smallest stable manifold among the ones in the family of Cantor rectangles $\mathcal{R}_{\delta_2}$. Thus, for all $\xi \in \Xi_i$

$$\int_{W_{\xi,A}} e^{-S_j^{-1}g}\nu = e^{-jP_*(T,g)}\int_{T^{-j}W_{\xi,A}} \nu \geqslant e^{-jP_*(T,g)}\bar{C}l_{\delta_2}^{\bar{C}_2(P_*(T,g)-\sup g)}.$$

Finally, let $D(R_i)$ be the smallest solid rectangle containing $R_i$. Since $\mu_g^W$ and $\nu$ are equivalent on $\mu_g$-a.e. $W \in \widehat{\mathcal{W}}^s$, we get for some $\xi \in \Xi_i$ such that $\nu(A \cap W_\xi) > 0$

$$\mu_g(e^{-S_j^{-1}g}\mathbb{1}_A) \geqslant \mu_g(e^{-S_j^{-1}g}\mathbb{1}_{A\cap D(R_i)}) \geqslant C''\nu(e^{-S_j^{-1}g}\mathbb{1}_{A\cap W_\xi}) \geqslant C'_{\delta_2}e^{-jP_*(T,g)}.$$

where we used (3.6.22) (with $\bar{E} = A$ and $E = A \cap W_\Xi$) for the second inequality. Since the family $\mathcal{R}_{\delta_2}$ is finite, this proves the lemma. □

**Proposition 3.6.18.** *If $g$ is a $(\mathcal{M}_0^1, \alpha_g)$-Hölder potential with $P_*(T,g) - \sup g > s_0 \log 2$, having SSP.1 and SSP.2, then the measure $\mu_g$ is the unique equilibrium state for $T$ under the potential $g$.*

*Proof.* Usually, given a known ergodic equilibrium state $\mu_g$, in order to prove uniqueness it suffices to check that for all $T$-invariant measure $\mu$ singular with respect to $\mu_g$, we have $h_\mu(T) + \mu(g) < h_{\mu_g}(T) + \mu_g(g)$ – see for example [KH95, Section 20.3]. This is the strategy we adopt.

Let $\mu$ be a $T$-invariant Borel probability measure, singular with respect to $\mu_g$, that is there exists a Borel set $F \subset M$ with $T^{-1}F = F$ and $\mu_g(F) = 0$ but $\mu(F) = 1$.

For each $n \in \mathbb{N}$, we consider the partition $\mathcal{Q}_n$ of maximal connected components of $M$ on which $T^{-n}$ is continuous. By [BD20, Lemma 3.2 and 3.3], $\mathcal{Q}_n$ is $\mathcal{M}_{-n}^0$ plus isolated points whose cardinality grows at most linearly with $n$. Thus $G_{2n} \subset \mathcal{Q}_{2n}$ for each $n$. Define $\tilde{B}_{2n} = \mathcal{Q}_{2n} \smallsetminus G_{2n}$. The set $\tilde{B}_{2n}$ contains $B_{2n}$ plus isolated points, and so its associated thermodynamic sum is bounded by the expression in Lemma 3.6.16 plus $\#\{\text{isolated points}\}e^{2n \sup g}$. Since $P_*(T,g) - \sup g > 0$, we have that $\frac{7}{4}P_*(T,g) + \frac{1}{4}\sup g > 2\sup g$, and thus the contribution of isolated points is small compared to the upperbound of Lemma 3.6.16.

By uniform hyperbolicity of $T$, the diameters of the elements of $T^{\lfloor n/2 \rfloor}\mathcal{Q}_n$ tend to zero as $n$ goes to infinity. This implies the following fact.

**Sublemma 3.6.19.** *For each $n \geqslant n_1$, there exists a finite union $\mathcal{C}_n$ of elements of $\mathcal{Q}_n$ such that*

$$\lim_{n\to+\infty} (\mu + \mu_g)(F \mathbin{\triangle} T^{-\lfloor n/2 \rfloor}\mathcal{C}_n) = 0\,.$$

*Proof.* The proof is essentially the same as [BD20, Sublemma 7.24] where the role of $\mu_*$ is played by $\mu_g$. Since notations are introduced in this proof, we write it down for completeness and latter use.

Let $\bar{\mu} = \mu + \mu_g$ and $\tilde{\mathcal{Q}}_n = T^{-\lfloor n/2 \rfloor}\mathcal{Q}_n$. For $\delta > 0$, by regularity of Radon measures, pick compact sets $K_1 \subset F$ and $K_2 \subset M \smallsetminus F$ such that $\max\{\bar{\mu}(F \smallsetminus K_1), \bar{\mu}((M \smallsetminus F) \smallsetminus K_2)\} < \delta$. Since $K_1$ and $K_2$ are disjoint and compact, we have $\eta = \eta_\delta := d(K_1, K_2) > 0$. If $\mathrm{diam}(\tilde{Q}) < \eta/2$, then either $\tilde{Q} \cap K_1 = \emptyset$ or $\tilde{Q} \cap K_2 = \emptyset$. Let $n_\delta$ be large enough so that the diameter of $\tilde{\mathcal{Q}}_k$ is smaller than $\eta_\delta/2$ for all $k \geqslant n_\delta$. Fix $n = 2n_\delta$ and set $\tilde{\mathcal{C}_n}$ to be the union of $\tilde{Q} \in \tilde{\mathcal{Q}}_n$ such that $\tilde{Q} \cap K_1 \neq \emptyset$. By construction, $K_1 \subset \tilde{\mathcal{C}}_n$ and $\tilde{\mathcal{C}}_n \cap K_2 = \emptyset$. Hence $\bar{\mu}(F \mathbin{\triangle} \tilde{\mathcal{C}}_n) \leqslant \delta + \bar{\mu}(K_1 \mathbin{\triangle} \tilde{\mathcal{C}}_n) \leqslant \delta + \bar{\mu}(M \smallsetminus (K_1 \cup K_2)) \leqslant 3\delta$. Defining $\mathcal{C}_n = T^{\lfloor n/2 \rfloor}\tilde{\mathcal{C}}_n$ completes the proof. □

Remark that since $T^{-1}F = F$, it follows that $(\mu + \mu_g)(\mathcal{C}_n \mathbin{\triangle} F)$ also tends to zero as $n \to +\infty$.

Since $\mathcal{Q}_{2n}$ is generating for $T^{2n}$, we have

$$h_\mu(T^{2n}) = h_\mu(T^{2n}, \mathcal{Q}_{2n}) \leqslant H_\mu(\mathcal{Q}_{2n}) = -\sum_{Q \in \mathcal{Q}_{2n}} \mu(Q)\log\mu(Q)\,.$$

Thus,

$$\begin{aligned} 2nP_\mu(T,g) = 2nh_\mu(T) + 2n\mu(g) = h_\mu(T^{2n}) + \mu(S_{2n}^{-1}g) \leqslant H_\mu(\mathcal{Q}_{2n}) + \mu(S_{2n}^{-1}g) \\ \leqslant \sum_{Q \in \mathcal{Q}_{2n}} \mu(Q)\Big(-\log\mu(Q) + S_{2n}^{-1}g(x_Q) + C_g\Big), \end{aligned}$$

where $x_Q \in Q$ and $C_g$ is the constant from Lemma 3.2.3.

Now, we want to distinguish elements of $\mathcal{Q}_{2n}$. From the proof of Sublemma 3.6.19, for each $n$, there exists a compact set $K_1(n)$ that defines $\tilde{\mathcal{C}}_n = T^{-\lfloor n/2 \rfloor}\mathcal{C}_n$, and satisfying $(\mu + \mu_g)(\cup_n K_1(n)) = (\mu + \mu_g)(F)$. We group elements $Q \in \mathcal{Q}_{2n} \subset \mathcal{Q}_n$ according to whether $T^{-n}Q \subset \tilde{\mathcal{C}}_n$ or $T^{-n}Q \cap \tilde{\mathcal{C}}_n = \emptyset$. This dichotomy is well defined because if $Q$ is not an isolated point, and if $T^{-n}Q \cap \tilde{\mathcal{C}}_n \neq \emptyset$, then $T^{-n}Q \in \mathcal{M}_{-n}^{n}$ is contained in an element of $\mathcal{M}_{-\lceil n/2 \rceil}^{\lfloor n/2 \rfloor}$ that intersect $K_1(n)$. Thus $Q \subset T^n\tilde{\mathcal{C}}_n = T^{\lceil n/2 \rceil}\mathcal{C}_n$ – the case where $Q$ is an isolated point is obvious. Therefore,

$$\begin{aligned}2nP_\mu(T,g) &\leqslant C_g + \sum_{Q\subset T^n\tilde{\mathcal{C}}_n} \mu(Q)\Big(-\log\mu(Q) + S_{2n}^{-1}g(x_Q)\Big)\\ &\qquad + \sum_{Q\in\mathcal{Q}_{2n}\smallsetminus T^n\tilde{\mathcal{C}}_n} \mu(Q)\Big(-\log\mu(Q) + S_{2n}^{-1}g(x_Q)\Big)\\ &\leqslant C_g + \frac{2}{e} + \mu(T^n\tilde{\mathcal{C}}_n)\log\left(\sum_{Q\subset T^n\tilde{\mathcal{C}}_n} e^{S_{2n}^{-1}g(x_Q)}\right)\\ &\qquad + \mu(M\smallsetminus T^n\tilde{\mathcal{C}}_n)\log\left(\sum_{Q\in\mathcal{Q}_{2n}\smallsetminus T^n\tilde{\mathcal{C}}_n} e^{S_{2n}^{-1}g(x_Q)}\right)\end{aligned}$$

where we used in the last line that the convexity of $x\log x$ implies that for all $p_j>0$ with $\sum_{j=1}^N p_j\leqslant 1$, and all $a_j\in\mathbb{R}$, we have (see [KH95, (20.3.5)])

$$\sum_{j=1}^N p_j(-\log p_j + a_j) \leqslant \frac{1}{e} + \sum_{j=1}^N p_j \log\sum_{i=1}^N e^{a_i}\,.$$

Then, since $-2nP_{\mu_g} = (\mu(T^n\tilde{\mathcal{C}}_n) + \mu(M\smallsetminus T^n\tilde{\mathcal{C}}_n))\log e^{-2nP_*(T,g)}$, we write for $n\geqslant n_1$

$$\begin{aligned}&2n(P_\mu(T,g) - P_{\mu_g}(T,g)) - \frac{2}{e} - C_g\\ &\leqslant \mu(T^{-n}\tilde{\mathcal{C}}_n)\log\left(\sum_{Q\subset T^n\tilde{\mathcal{C}}_n} e^{S_{2n}^{-1}g(x_Q)-2nP_*(T,g)}\right)\\ &\quad + \mu(M\smallsetminus T^{-n}\tilde{\mathcal{C}}_n)\log\left(\sum_{Q\in\mathcal{Q}_{2n}\smallsetminus T^n\tilde{\mathcal{C}}_n} e^{S_{2n}^{-1}g(x_Q)-2nP_*(T,g)}\right)\\ &\leqslant \mu(\mathcal{C}_n)\log\left(\sum_{\substack{Q\subset T^n\tilde{\mathcal{C}}_n\\ Q\in G_{2n}}} e^{S_{2n}^{-1}g(x_Q)-2nP_*(T,g)} + \sum_{\substack{Q\subset T^n\tilde{\mathcal{C}}_n\\ Q\in\tilde{B}_{2n}}} e^{S_{2n}^{-1}g(x_Q)-2nP_*(T,g)}\right)\\ &\quad + \mu(M\smallsetminus\mathcal{C}_{2n})\log\left(\sum_{Q\in G_{2n}\smallsetminus T^n\tilde{\mathcal{C}}_n} e^{S_{2n}^{-1}g(x_Q)-2nP_*(T,g)} + \sum_{Q\in\tilde{B}_{2n}\smallsetminus T^n\tilde{\mathcal{C}}_n} e^{S_{2n}^{-1}g(x_Q)-2nP_*(T,g)}\right)\end{aligned} \tag{3.6.24}$$

where we used that $\mathcal{Q}_{2n} = G_{2n}\sqcup\tilde{B}_{2n}$. By Lemma 3.6.16 (and the remark concerning the contribution of isolated points), both sums over elements of $\tilde{B}_{2n}$ are bounded by $Ce^{-\frac{1}{4}n(P_*(T,g)-\sup g)}$.

It remains to estimate both sums over elements of $G_{2n}$. To do so, we want to use Lemma 3.6.17, that is for each $Q\in G_{2n}$, we want to assign a set $\bar{E}$ satisfying the assumptions of the lemma. Let $Q\in G_{2n}$. Thus $Q\notin B^0_{-2n}$, and so there exists $0\leqslant j\leqslant\lfloor n/2\rfloor$ such that $T^{-j}Q\subset E_j\in\mathcal{M}^0_{2n+j}$ with $\operatorname{diam}^u(E_j)\geqslant\delta_2$. Also, since $T^{-2n}Q\notin B_0^{2n}$, there exists $0\leqslant k\leqslant\lfloor n/2\rfloor$ such that $T^{-2n+k}Q\subset\tilde{E}_k\in\mathcal{M}_0^{2n-k}$ with $\operatorname{diam}^s(\tilde{E}_k)\geqslant\delta_2$. Thus, both $\tilde{E}_k\in\mathcal{M}_0^{2n-k}$ and $T^{-2n+j+k}E_j\in\mathcal{M}_{-k}^{2n-j-k}$ contain $T^{-2n+k}Q$. In particular, there exists $\tilde{E}\in\mathcal{M}_0^{2n-j-k}$ containing both $\tilde{E}_k$ and $T^{-2n+j+k}E_j$. Let $\bar{E} = T^{2n-j-k}\tilde{E}\in\mathcal{M}^0_{-2n+j+k}$. Notice that by construction $E_j \subset \bar{E}$ and $\tilde{E}_k \subset T^{-2n+j+k}\bar{E}$, therefore $\bar{E}$ satisfies $\mathrm{diam}^u(\bar{E}) \geqslant \delta_2$ and $\mathrm{diam}^s(T^{-2n+j+k}\bar{E}) \geqslant \delta_2$, the assumption from Lemma 3.6.17. Thus,

$$\mu_g(e^{-S^{-1}_{2n-j-k}g}\mathbb{1}_{\bar{E}}) \geqslant C_{\delta_2} e^{-(2n-j-k)P_*(T,g)} .$$

We call $(\bar{E}, j, k)$ an *admissible triple* for $Q \in G_{2n}$ if $0 \leqslant j, k \leqslant \lfloor n/2 \rfloor$ and $\bar{E} \in \mathcal{M}^0_{-2n+j+k}$, with $T^{-j}Q \subset \bar{E}$ and $\min\{\mathrm{diam}^u(\bar{E}), \mathrm{diam}^s(T^{-2n+j+k}\bar{E})\} \geqslant \delta_2$. By the above construction, such admissible triples always exist, but there may be many associated to a given $Q \in G_{2n}$. However, we can define the unique *maximal triple* for $Q$ by taking first the maximum $j$, and then the maximum $k$ over all admissible triples for $Q$.

Let $\mathcal{E}_{2n}$ be the set of maximal triples obtained in this way from elements of $G_{2n}$. For $(\bar{E}, j, k) \in \mathcal{E}_{2n}$, let $\mathcal{A}_M(\bar{E}, j, k)$ denote the set of $Q \in G_{2n}$ for which the maximal triple is $(\bar{E}, j, k)$. The importance of the set $\mathcal{E}_{2n}$ lies in [BD20, Sublemma 7.25], which we state, and prove, as follows for completeness.

**Sublemma 3.6.20.** *Suppose that* $(\bar{E}_1, j_1, k_1)$, $(\bar{E}_2, j_2, k_2) \in \mathcal{E}_{2n}$ *with* $j_2 \geqslant j_1$ *and* $(\bar{E}_1, j_1, k_1) \neq (\bar{E}_2, j_2, k_2)$. *Then* $T^{-(j_2-j_1)}\bar{E}_1 \cap \bar{E}_2 = \emptyset$.

*Proof.* By contradiction, let $(\bar{E}_1, j_1, k_1)$, $(\bar{E}_2, j_2, k_2) \in \mathcal{E}_{2n}$ with $j_2 \geqslant j_1$, $(\bar{E}_1, j_1, k_1) \neq (\bar{E}_2, j_2, k_2)$ and $T^{-(j_2-j_1)}\bar{E}_1 \cap \bar{E}_2 \neq \emptyset$. Notice that $T^{-(j_2-j_1)}\bar{E}_1 \in \mathcal{M}^{j_2-j_1}_{-2n+j_2+k_1}$ while $\bar{E}_2 \in \mathcal{M}^0_{-2n+j_2+k_2}$.

Consider first the case $k_1 \leqslant k_2$. Therefore $T^{-(j_2-j_1)}\bar{E}_1 \subset E_2$. In particular, any element $Q \in \mathcal{A}_M(\bar{E}_1, j_1, k_1)$ satisfies $T^{-j_2}Q \subset \bar{E}_2$, and so $Q \in \mathcal{A}_M(\bar{E}_2, j_2, k_2)$, a contradiction.

Consider now the case $k_1 > k_2$. Therefore $T^{-(j_2-j_1)}\bar{E}_1$ and $\bar{E}_2$ are both contained in an element $\bar{E}' \in \mathcal{M}^0_{-2n+j_2+k_1}$. Since $\bar{E}_2 \subset \bar{E}'$, we have that $\mathrm{diam}^u(\bar{E}') \geqslant \delta_2$. Also, since $T^{-2n+j_1+k_1}\bar{E}_1 \subset T^{-2n+j_2+k_1}\bar{E}'$, we have that $\mathrm{diam}^s(T^{-2n+j_2+k_1}\bar{E}') \geqslant \delta_2$. Note that if $Q \in \mathcal{A}_M(\bar{E}_1, j_1, k_1) \cup \mathcal{A}_M(\bar{E}_2, j_2, k_2)$, then $(\bar{E}', j_2, k_1)$ is an admissible triple for $Q$. Thus, if $j_1 = j_2$, then $\bar{E}' = \bar{E}_1$. For $Q \in \mathcal{A}_M(\bar{E}_2, j_2, k_2)$, then $Q \subset \bar{E}_1$ and so $(\bar{E}_1, j_1, k_1)$ is an admissible triple for $Q$, which contradicts the maximality of $(\bar{E}_2, j_2, k_2)$ since $k_1 > k_2$. Similarly, if $j_2 > j_1$, then for $Q \in \mathcal{A}_M(\bar{E}_1, j_1, k_1)$, the triple $(\bar{E}', j_2, k_1)$ is admissible for $Q$, which contradicts the maximality of $(\bar{E}_1, j_1, k_1)$. □

We now prove that if $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) \neq \emptyset$, then $\mathcal{A}_M(\bar{E}, j, k) \subset T^n\tilde{\mathcal{C}}_n$ and $\bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n$. Let $Q \in \mathcal{A}_M(\bar{E}, j, k)$ be such that $Q \cap T^n\tilde{\mathcal{C}}_n \neq \emptyset$. Then, by definition of $(\bar{E}, j, k)$, $T^{-n}Q \subset T^{-n+j}\bar{E} \in \mathcal{M}^{n-j}_{-n+k}$. Since $0 \leqslant j, k \leqslant \lfloor n/2 \rfloor$, there exists $E' \in \mathcal{M}^{\lfloor n/2 \rfloor}_{-\lfloor n/2 \rfloor}$ such that $T^{-n+j}\bar{E} \subset E'$. In particular, we have $E' \in \tilde{\mathcal{Q}}_n$ and $E' \cap \tilde{\mathcal{C}}_n \neq \emptyset$. Thus, by construction of $\tilde{\mathcal{C}}_n$, we have $\tilde{\mathcal{C}}_n \supset E' \supset T^{-n+j}\bar{E} \supset T^{-n}Q$. In particular, we get $Q \subset T^n\tilde{\mathcal{C}}_n$, and thus $\mathcal{A}_M(\bar{E}, j, k) \subset T^n\tilde{\mathcal{C}}_n$. We also get $\bar{E} \subset T^{n-j}\tilde{\mathcal{C}}_n$.

On the other hand, we prove that if $T^n\tilde{\mathcal{C}}_n \cap \mathcal{A}_M(\bar{E}, j, k) = \emptyset$, then $\mathcal{A}_M(\bar{E}, j, k) \subset M \setminus T^n\tilde{\mathcal{C}}_n$ and $T^{-n+j}\bar{E} \subset M \setminus \tilde{\mathcal{C}}_n$. Let $Q \in \mathcal{A}_M(\bar{E}, j, k)$. Then, by assumption, $T^{-n}Q \cap \tilde{\mathcal{C}}_n = \emptyset$. As above, there exists $E' \in \mathcal{M}^{\lfloor n/2 \rfloor}_{-\lfloor n/2 \rfloor}$ containing both $T^{-n}Q$ and $T^{-n+j}\bar{E}$. In particular, $E' \in \tilde{\mathcal{Q}}_n$ and $E' \cap \tilde{\mathcal{C}}_n = \emptyset$. By construction of $\tilde{\mathcal{C}}_n$, we get that $E' \in M \setminus \tilde{\mathcal{C}}_n$. Thus $Q \in M \setminus T^n\tilde{\mathcal{C}}_n$, and so $\mathcal{A}_M(\bar{E}, j, k) \subset M \setminus T^n\tilde{\mathcal{C}}_n$. Also, $T^{-n+j}\bar{E} \subset M \setminus \tilde{\mathcal{C}}_n$.

The only last step we have to do before estimating the sums over $G_{2n}$ is to prove that for each $(\bar{E}, j, k) \in \mathcal{E}_{2n}$, we have

$$\sum_{Q \in \mathcal{A}_M(\bar{E}, j, k)} |e^{S^{-1}_{2n}g}|_{C^0(Q)} \leqslant Ce^{(j+k)P_*(T,g)} |e^{S^{-1}_{2n-j-k}g}|_{C^0(\bar{E})} \tag{3.6.25}$$

where $C > 0$ is a constant depending only on the potential $g$. To do so, notice that if $Q \in \mathcal{A}_M(\bar{E}, j, k)$, then by construction, $T^{-j}Q \subset \bar{E}$. Thus $T^{-n}Q \in \mathcal{M}_{-n}^{n}$ is a subset of $T^{-(n-j)}\bar{E} \in \mathcal{M}_{-n+k}^{n-j}$. Decomposing $T^{-n}Q = Q_- \cap Q_+$ with $Q_- \in \mathcal{M}_{-n}^0$ and $Q_+ \in \mathcal{M}_0^n$, and $T^{-(n-j)}\bar{E} = E_- \cap E_+$ with $E_- \in \mathcal{M}_{-n+k}^0$ and $E_+ \in \mathcal{M}_0^{n-j}$, we see that $Q_- \subset E_-$ and $Q_+ \subset E_+$. Thus

$$\begin{aligned}
\sum_{Q \in \mathcal{A}_M(\bar{E},j,k)} |e^{S_{2n}^{-1} g}|_{C^0(Q)} &= \sum_{Q \in \mathcal{A}_M(\bar{E},j,k)} |e^{S_{2n}^{-1} g \circ T^n}|_{C^0(T^{-n}Q)} \\
&\leqslant \sum_{\substack{Q_- \in \mathcal{M}_{-n}^0 \\ Q_- \subset E_-}} \sum_{\substack{Q_+ \in \mathcal{M}_0^n \\ Q_+ \subset E_+}} |e^{S_n^{-1} g + S_n g \circ T}|_{C^0(Q_- \cap Q_+)} \\
&\leqslant \sum_{\substack{Q_- \in \mathcal{M}_{-n}^0 \\ Q_- \subset E_-}} |e^{S_n^{-1} g}|_{C^0(Q_-)} \sum_{\substack{Q_+ \in \mathcal{M}_0^n \\ Q_+ \subset E_+}} |e^{S_n g \circ T}|_{C^0(Q_+)} \\
&\leqslant \sum_{\substack{Q_- \in \mathcal{M}_{-n}^0 \\ Q_- \subset E_-}} |e^{S_n^{-1} g \circ T^{n-k}}|_{C^0(T^{-n+k}Q_-)} \sum_{\substack{Q_+ \in \mathcal{M}_0^n \\ Q_+ \subset E_+}} |e^{S_n g \circ T \circ T^{-(n-j)}}|_{C^0(T^{n-j}Q_+)} .
\end{aligned}$$

Now, notice that $T^{-n+k}Q_- \in \mathcal{M}_{-k}^{n-k}$ is a subset of $T^{-n+k}E_- \in \mathcal{M}_0^{n-k}$. Thus $T^{-n+k}Q_-$ must be of the form $\tilde{Q}_- \cap T^{-n+k}E_-$ for some $\tilde{Q}_- \in \mathcal{M}_{-k}^0$. Similarly, $T^{n-j}Q_+$ must be of the form $\tilde{Q}_+ \cap T^{n-j}E_+$ for some $\tilde{Q}_+ \in \mathcal{M}_0^j$. Thus

$$\begin{aligned}
&\sum_{Q \in \mathcal{A}_M(\bar{E},j,k)} |e^{S_{2n}^{-1} g}|_{C^0(Q)} \\
&\leqslant \sum_{\tilde{Q}_- \in \mathcal{M}_{-k}^0} |e^{S_n^{-1} g \circ T^{n-k}}|_{C^0(\tilde{Q}_- \cap T^{-n+k}E_-)} \sum_{\tilde{Q}_+ \in \mathcal{M}_0^j} |e^{S_n g \circ T \circ T^{-(n-j)}}|_{C^0(\tilde{Q}_+ \cap T^{n-j}E_+)} \\
&\leqslant \sum_{\tilde{Q}_- \in \mathcal{M}_{-k}^0} |e^{S_k^{-1} g}|_{C^0(\tilde{Q}_-)} |e^{S_{2n-j-k}^{-1} g - S_{n-j}^{-1} g}|_{C^0(T^{n-j}E_-)} \sum_{\tilde{Q}_+ \in \mathcal{M}_0^j} |e^{S_j g \circ T}|_{C^0(\tilde{Q}_+)} |e^{S_{n-j}^{-1} g}|_{C^0(T^{n-j}E_+)} .
\end{aligned}$$

Now, using Lemma 3.2.3, the supermultiplicativity from Lemma 3.3.9 and the exact exponential growth from Proposition 3.3.10, we get the upper bound (3.6.25) with $C = 2\, C_g\, e^{\sup g - \inf g}$.

We can now estimate the sums over elements of $G_{2n}$.

$$\begin{aligned}
\sum_{\substack{Q \in G_{2n} \\ Q \subset T^n \tilde{\mathcal{C}}_n}} e^{S_{2n}^{-1} g(x_Q) - 2nP_*(T,g)} &\leqslant \sum_{\substack{(\bar{E},j,k) \in \mathcal{E}_{2n} \\ \bar{E} \subset T^{n-j} \tilde{\mathcal{C}}_n}} \sum_{Q \in \mathcal{A}_M(\bar{E},j,k)} e^{S_{2n}^{-1} g(x_Q) - 2nP_*(T,g)} \\
&\leqslant \sum_{\substack{(\bar{E},j,k) \in \mathcal{E}_{2n} \\ \bar{E} \subset T^{n-j} \tilde{\mathcal{C}}_n}} C e^{-(2n-j-k)P_*(T,g)} |e^{S_{2n-j-k}^{-1} g}|_{C^0(\bar{E})} \\
&\leqslant \sum_{\substack{(\bar{E},j,k) \in \mathcal{E}_{2n} \\ \bar{E} \subset T^{n-j} \tilde{\mathcal{C}}_n}} C C_{\delta_2}^{-1} \mu_g(e^{-S_{2n-j-k}^{-1} g} \mathbb{1}_{\bar{E}}) |e^{S_{2n-j-k}^{-1} g}|_{C^0(\bar{E})} \\
&\leqslant C C_{\delta_2}^{-1} C_g \sum_{\substack{(\bar{E},j,k) \in \mathcal{E}_{2n} \\ \bar{E} \subset T^{n-j} \tilde{\mathcal{C}}_n}} \mu_g(\bar{E}) \leqslant C C_{\delta_2}^{-1} C_g \sum_{\substack{(\bar{E},j,k) \in \mathcal{E}_{2n} \\ \bar{E} \subset T^{n-j} \tilde{\mathcal{C}}_n}} \mu_g(T^{-n+j}\bar{E}) \\
&\leqslant C'\, \mu_g(\tilde{\mathcal{C}}_n)
\end{aligned}$$

where $C' = CC_{\delta_2}^{-1}C_g$.

Similarly,

$$\begin{aligned}\sum_{Q\in G_{2n}\setminus T^n\tilde{\mathcal{C}}_n} e^{S_{2n}^{-1}g(x_Q)-2nP_*(T,g)} &\leqslant \sum_{\substack{(\bar{E},j,k)\in\mathcal{E}_{2n}\\ \bar{E}\subset M\setminus T^{n-j}\tilde{\mathcal{C}}_n}} \sum_{Q\in\mathcal{A}_M(\bar{E},j,k)} e^{S_{2n}^{-1}g(x_Q)-2nP_*(T,g)}\\
&\leqslant \sum_{\substack{(\bar{E},j,k)\in\mathcal{E}_{2n}\\ \bar{E}\subset M\setminus T^{n-j}\tilde{\mathcal{C}}_n}} Ce^{-(2n-j-k)P_*(T,g)}|e^{S_{2n-j-k}^{-1}g}|_{C^0(\bar{E})}\\
&\leqslant \sum_{\substack{(\bar{E},j,k)\in\mathcal{E}_{2n}\\ \bar{E}\subset M\setminus T^{n-j}\tilde{\mathcal{C}}_n}} CC_{\delta_2}^{-1}\mu_g(e^{-S_{2n-j-k}^{-1}g}\mathbb{1}_{\bar{E}})|e^{S_{2n-j-k}^{-1}g}|_{C^0(\bar{E})}\\
&\leqslant CC_{\delta_2}^{-1}C_g \sum_{\substack{(\bar{E},j,k)\in\mathcal{E}_{2n}\\ \bar{E}\subset M\setminus T^{n-j}\tilde{\mathcal{C}}_n}} \mu_g(\bar{E}) \leqslant CC_{\delta_2}^{-1}C_g \sum_{\substack{(\bar{E},j,k)\in\mathcal{E}_{2n}\\ \bar{E}\subset M\setminus T^{n-j}\tilde{\mathcal{C}}_n}} \mu_g(T^{-n+j}\bar{E})\\
&\leqslant C'\,\mu_g(M\setminus\tilde{\mathcal{C}}_n)\end{aligned}$$

Putting these bounds together allows us to complete our estimate of (3.6.24),

$$\begin{aligned}2n(P_\mu(T,g)-P_{\mu_g}(T,g)) - \frac{2}{e} - C_g \leqslant\ & \mu(\mathcal{C}_n)\log\Big(C'\mu_g(\mathcal{C}_n) + Ce^{-\frac{1}{4}n(P_*(T,g)-\sup g)}\Big)\\
&+ \mu(M\setminus\mathcal{C}_n)\log\Big(C'\mu_g(M\setminus\mathcal{C}_n) + Ce^{-\frac{1}{4}n(P_*(T,g)-\sup g)}\Big).\end{aligned}$$

Since by Sublemma 3.6.19 $\mu(\mathcal{C}_n)$ tends to 1 as $n\to+\infty$, while $\mu_g(\mathcal{C}_n)$ tends to 0 as $n\to+\infty$, the limit of the right-hand side tends to $-\infty$. This yields a contradiction unless $P_\mu(T,g) < P_{\mu_g}(T,g)$. □

## 3.7 The Billiard Flow

Throughout this section, we see the billiard flow $\phi_t$ as the vertical flow in the space

$$\tilde{\Omega} = \{(x,t)\in M\times\mathbb{R} \mid 0\leqslant t\leqslant\tau(x)\}/\sim,$$

where the equivalence relation is defined by $(x,\tau(x))\sim(T(x),0)$. In other words, we see $\phi_t$ as the suspension flow over $T$ under the return time $\tau$. Furthermore, transporting the Euclidean metric on $\mathcal{Q}\times\mathbb{S}^1$ onto $\tilde{\Omega}$, the flow $\phi_t$ is uniformly hyperbolic.

**Proposition 3.7.1.** *Let $g$ be a $(\mathcal{M}_0^1,\alpha_g)$-Hölder potential such that $P_*(T,g)-\sup g > s_0\log 2$, with SSP.1 and SSP.2. Let $\bar{\mu}_g = (\mu_g(\tau))^{-1}\mu_g\otimes\lambda$, where $\lambda$ is the Lebesgue measure. Then $(\phi_t,\bar{\mu}_g)$ is a K-system.*

*Proof.* The ergodicity of $(\phi_t,\bar{\mu}_g)$ follows directly from the ergodicity of $(T,\mu_g)$ proved in Proposition 3.6.12.

To prove the K-mixing, we follow closely the method used in Sections 6.9, 6.10 and 6.11 from [CM06]. In fact, replacing $\mu$ and $\mu_\Omega$ with $\mu_g$ and $\bar{\mu}_g$ throughout these sections, we only have to check that [CM06, Exercise 6.35] is still true in order to apply verbatim the arguments. This is what we prove here.

To do so, we first need to recall some of the construction done in [CM06, Section 6.9]. If $x_1$ and $x_3$ are two nearby points in $M$ such that

$$\{x_2\} := W^u(x_1)\cap W^s(x_3)\neq\emptyset\,,\quad \{x_4\} := W^s(x_1)\cap W^u(x_3)\neq\emptyset\,, \tag{3.7.1}$$

we then construct the 4-loop $Y_1$, $Y_2$, $Y_3$, $Y_4$, $Y_5 \in \Omega$ as follow. Let $Y_1 = X_1 = (x_1, t)$ and $X_3 = (x_3, t)$. Define

$$Y_2 = W^u(Y_1) \cap W^{ws}_{loc}(X_3), \quad Y_3 = W^s(Y_2) \cap W^{wu}_{loc}(X_3),$$
$$Y_4 = W^u(Y_3) \cap W^{ws}_{loc}(X_1), \quad Y_5 = W^s(Y_4) \cap W^{wu}_{loc}(X_1),$$

where $W^u$ and $W^s$ are unstable and stable manifolds for the flow, and $W^{wu}_{loc}$ and $W^{ws}_{loc}$ are local weak unstable and local weak stable manifolds for the flow. We always assume that this construction stays under the ceiling function $\tau$. Actually, as proven in [CM06, Lemma 6.40] there exists $\sigma$ such that $Y_5 = \phi_\sigma(Y_1)$, with $|\sigma| = \mu_{\text{SRB}}(K)$ where $K$ is the rectangle in $M$ with corners $x_1$, $x_2$, $x_3$, $x_4$. Thus the 4-loops are always open.

For $x \in M$, let $\mathcal{L}_x = \{\phi_t(x) \mid 0 < t < \tau(x)\}$. Then the partition $\{\mathcal{L}_x \mid x \in M\}$ of $\tilde{\Omega}$ is measurable and the conditional measures of $\bar{\mu}_g$ on $\mathcal{L}_x$ are uniform. Call $\lambda_x$ the Lebesgue probability measure on $\mathcal{L}_x$. Let $D \subset \Omega$ be such that $\bar{\mu}_g(D) = 1$ and let $E_1 = \{x \in M \mid \lambda_x(\mathcal{L}_x \smallsetminus D) = 0\}$. Clearly, $\mu_g(E_1) = 1$.

We call a point $x_1 \in E_1$ *rich* if for any $\varepsilon > 0$ there exists another point $x_3 \in E_1$ such that $0 < d(x_1, x_3) < \varepsilon$ and (3.7.1) holds with $x_2$ and $x_4 \in E_1$. Denote $E_2 \subset E_1$ the set of rich points.

The analogous of [CM06, Exercise 6.35] is to prove that $\mu_g(E_2) = 1$. Let $\{R_j\}_{j \geqslant 1}$ be the cover of $M^{\text{reg}}$ into Cantor rectangles (discarding the ones with zero $\mu_g$-measure). Let $R$ be one of those Cantor rectangle and denote $\mu_R$ the conditional measure of $\mu_g$ on $R$. It is enough to prove that $\mu_R(E_2) = 1$. Since $\mu_g(E_1) = 1$ we have that $\mu_R(E_1) = 1$. Furthermore, since the partition of $R$ into stable manifolds is measurable, we can disintegrate $\mu_R$ with respect to this partition, with conditional measure $\mu^W_s$ on $W \in R \cap \mathcal{W}^s$. It follows that for $\mu_R$-a.e. point $x \in E_1 \cap R$, if $W = W(x) \in \mathcal{W}^s$ contains $x$ then $\mu^W_s(W \cap E_1) = 1$. Similarly, for $\mu_R$-a.e. point $x \in E_1 \cap R$, if $W = W(x) \in \mathcal{W}^u$ contains $x$ then $\mu^W_u(W \cap E_1) = 1$, where $\mu^W_u$ is the conditional measure on $W$ in the disintegration of $\mu_R$ with respect to the measurable partition $R \cap \mathcal{W}^u$ of $R$. Then $\mu_R(E_R) = 1$, where $E_R$ denotes the set of points $x$ in $R$ such that both stable and unstable conditional measure on leaves containing $x$ give measure 1 to $E_1$.

Let $E^R_2 \subset E_2$ be the set of rich points $x_1$ such that $x_3$ belongs to $R \cap E_1$ (and therefore $x_2$ and $x_4$ also belong to $R \cap E_1$ by the properties of a Cantor rectangle). By contradiction, assume that $\mu_R(E^R_2) \neq 1$. Define the sets

$$C^R_2 = \{x_1 \in E_1 \cap R \mid \exists \varepsilon > 0, \forall x_3 \in E_1 \cap R, \text{ if } 0 < d(x_1, x_3) < \varepsilon \text{ then } x_2 \notin E_1 \cap R\},$$
$$C^R_4 = \{x_1 \in E_1 \cap R \mid \exists \varepsilon > 0, \forall x_3 \in E_1 \cap R, \text{ if } 0 < d(x_1, x_3) < \varepsilon \text{ then } x_4 \notin E_1 \cap R\}.$$

Note that we don't have to introduce condition (3.7.1) in these definitions since it is automatically satisfied by the construction of Cantor rectangles. Thus, we have $(E_1 \cap R) \smallsetminus E^R_2 = C^R_2 \cup C^R_4$, so that $\mu_R(C^R_2 \cup C^R_4) > 0$. Assume first that $\mu_R(C^R_2) > 0$. Define the family of sets

$$C^R_{2,n} = \{x_1 \in C^R_2 \mid \varepsilon \geqslant \tfrac{1}{n}\}.$$

Since $\bigcup_{n \geqslant 1} C^R_{2,n} = C^R_2$ is an increasing union, there is some $n$ such that $\mu_R(C^R_{2,n}) > 0$. Let $x_1 \in C^R_{2,n} \cap E_R$ and $W \in \mathcal{W}^u$ be such that $x_1 \in W$. Let $x_3 \in E_1 \cap R \cap E_R$ be such that $0 < d(x_1, x_3) < \frac{1}{n}$. Let $W_0 \in \mathcal{W}^u$ be the unstable manifold containing $x_3$. By construction of $E_R$, we have $\mu^{W_0}_u(W_0 \cap E_1) = 1$, and since $\mu^{W_0}_u$ have support $W_0$ (otherwise, $\mu_g$ would not have total support because of the absolute continuity of the holonomy), in fact we have that

$$\mu_u^{W_0}(W_0 \cap E_1 \cap B(x_1, \tfrac{1}{n})) > 0.$$

Thus $\mu_u^W(\Theta_W(W_0 \cap E_1 \cap B(x_1, \frac{1}{n}))) > 0$. Now, if $\tilde{x}_3 \in W_0 \cap E_1 \cap B(x_1, \frac{1}{n})$, then $\tilde{x}_2 \notin E_1$. In other words, $E_1 \cap \Theta_W(W_0 \cap E_1 \cap B(x_1, \frac{1}{n})) = \emptyset$. Since $x_1 \in E_R$, we have that $\mu_u^W(W \cap E_1) = 1$, so that $\mu_u^W(\Theta_W(W_0 \cap E_1 \cap B(x_1, \frac{1}{n}))) = 0$, a contradiction. Thus $E_R \cap C_{2,n}^R = \emptyset$, so that $\mu_R(C_2^R) = 0$. We proceed similarly, exchanging the role of $\mathcal{W}^s$ and $\mathcal{W}^u$, in order to prove that $\mu_R(C_4^R) = 0$. Finally, we get that $\mu_R(E_2^R) = 1$, the contradiction closing the proof. □

**Proposition 3.7.2.** *Under the assumptions of Proposition 3.7.1, $(\phi_t, \bar{\mu}_g)$ is Bernoulli.*

*Proof.* The idea of the proof is to bootstrap from the K-mixing following the techniques of [CH96] with modifications similar to those in [BD20, Proposition 7.19]. The proof in [CH96] proceeds in two steps.

*Step 1. Construction of $\delta$-regular coverings.* Given $\delta > 0$, the idea is to cover $\tilde{\Omega}$, up to a set of $\bar{\mu}_g$-measure at most $\delta$, by small Cantor boxes – essentially a set of the form Cantor rectangle times interval – such that $\bar{\mu}_g$ restricted to each Cantor box is arbitrarily close to a product measure. The basis of the boxes will be very similar to the covering $\{R_i\}_{i\in\mathbb{N}}$ from Lemma 3.6.9, however, some adjustments must be made in order to guarantee uniform properties of the Jacobian of the relevant holonomy map.

Above a Cantor rectangle $R$ with $\mu_g(R) > 0$, we construct a Cantor box $B$ following the construction of *P-sets* from [OW73, Section 3]. Let $W_1^s$ and $W_2^s$ be the stable sides of the smallest solid rectangle $D(R)$ containing $R$. Let $W$ be a stable manifold for $\phi_t$ projecting on $W_1^s$ through the map $P_- : (x,t) \in \Omega \mapsto x \in M$ if $t < \tau(x)$, and being such that $W \subset \tilde{\Omega}_0 := \{(x,t) \mid 0 < t < \tau(x)\}$. Consider the set $W_R \subset W$ of points $(x,t) \in W$ such that $x \in R$. Let $t_0$ be small enough so that $S = \bigcup_{t=0}^{t_0} \phi_t(W_R) \subset \Omega_0$. Now, $B_0$ is obtained by moving $S$ along the unstable manifolds of $\phi_t$ to another surface of that type, spanned by $W_2^s$. That is, for each $(x,t) \in S$, take the unstable manifold $W(x,t)$ of $\phi_t$ passing by $(x,t)$, and projecting on the unstable manifold for $T$ passing by $x \in R$. Let $B_0 = \bigcup_{(x,t)\in S} W(x,t)$ and let $B \subset B_0$ be the set of points $(x,t) \in B_0$ such that $x \in R$. Notice that, up to subdividing $R$ into smaller rectangles taking a smaller $t_0$, we can assume that $B \subset \tilde{\Omega}_0$. Thus, by construction, the set $B$ has the property that for all $x, y \in B$, the local unstable manifold of $x$ and the local weakly stable manifold of $y$ intersect each other at a single point which lies in $B$. This is the crucial property of what Ornstein and Weiss, in [OW73], called a *rectangle*.

Since $\mu_g(R) > 0$, we have $\bar{\mu}_g(B) = t_0 \mu_g(R) > 0$, so that the conditional measure $\bar{\mu}_B$ of $\bar{\mu}_g$ restricted to $B$ makes sense. Now, we want to equip $B$ with a product measure, absolutely continuous with respect to $\bar{\mu}_B$. We proceed as follows. Since the partition of $B$ into unstable manifolds is measurable, we can disintegrate $\bar{\mu}_B$ into conditional measures $\bar{\mu}^{W_\xi}$, on $W_\xi \cap B$ with $\xi \in Z_\phi$, and a factor measure $\hat{\bar{\mu}}$ on the set $Z_\phi$ parametrizing the unstable manifolds of $B$. Fix a point $z \in B$, and consider $B$ as the product of $W^u(z) \cap B$ with $W^{ws}(z) \cap B$, where $W^u(z)$ is the local unstable manifold of $z$ and $W^{ws}(z)$ is the local weak stable manifold of $z$. Define $\bar{\mu}_B^p = \bar{\mu}^{W^u(z)} \otimes \hat{\bar{\mu}}$, and note that we can view $\hat{\bar{\mu}}$ as inducing a measure on $W^{ws}(z)$. We still have to prove that $\bar{\mu}_B^p \ll \bar{\mu}_B$.

Similarly, let $\mu_R$ be the conditional measure of $\mu_g$ restricted to $R$. Since the partition into unstable manifolds $W_\xi$, $\xi \in Z$, is measurable, we can disintegrate $\mu_R$ into the conditional measures $\mu^W$ on $W \cap R$ and a factor measure $\hat{\mu}$ on $Z$. We want to relate the disintegration $\bar{\mu}_B$ with the one of $\mu_R$. Notice that we can view $Z_\phi$ as the set $Z \times [0, t_0]$, where $Z$ parametrize the set of unstable manifolds of $R$ through the map associating $\xi_\phi \in Z_\phi$ with the pair $(\xi, t)$ where $\xi \in Z$ is such that $P_-(W_{\xi_\phi}) = W_\xi \subset D(R)$ and $t$ is the value in the definition of $S$ where $W_{\xi_\phi}$ and $S$ intersect. Considering sets $A \subset B$ of the form $A = P_-(A) \times [t_-, t_+]$, we get that

$$\int_{\xi_\phi \in Z_\phi} \bar{\mu}^{W_{\xi_\phi}}(A)\,\mathrm{d}\hat{\bar{\mu}}(\xi_\phi) = \bar{\mu}_B(A) = \int_{t_-}^{t_+} \mu_R(P_-(A))\,\mathrm{d}t = \int_{t_-}^{t_+} \int_{\xi \in Z} \mu^{W_\xi}(P_-(A))\,\mathrm{d}\hat{\mu}(\xi)\,\mathrm{d}t.$$

Thus, we can identify $\bar{\mu}^{W_{\xi_\phi}}$ with $\mu^{P_-(W_{\xi_\phi})}$, and $\mathrm{d}\hat{\bar{\mu}}$ with $\mathrm{d}\hat{\mu}\mathrm{d}t$. From this identifications, we deduce that the projection map $P_{W,-}$ from some $W$ to $P_-(W)$, and its inverse are absolutely continuous. The absolute continuity of the holonomy map $\bar{\Theta}_W$ between unstable manifolds $W_0$ and $W$ in $B$ thus follows directly from the absolute continuity of the holonomy map between unstable manifolds in $R$ since $\bar{\Theta}_W = P_{W,-}^{-1} \circ \theta_{P_-(W)} \circ P_{W,-}$. This implies that $\bar{\mu}_B^p$ is absolutely continuous with respect to $\bar{\mu}_B$, and thus, also with respect to $\bar{\mu}_g$. The following definition is taken from [CH96].

**Definition 3.7.3.** *For $\delta > 0$, a $\delta$-regular covering of $\Omega$ is a finite collection $\mathfrak{B}$ of disjoint Cantor boxes for which*[17]*,*

*a)* $\bar{\mu}_g(\bigcup_{B \in \mathfrak{B}} B) \geqslant 1 - \delta$*.*

*b) Every $B \in \mathfrak{B}$ satisfies $\left|\frac{\bar{\mu}_B^p(B)}{\bar{\mu}_g(B)} - 1\right| < \delta$. Moreover, there exists $G \subset B$, with $\bar{\mu}_g(G) > (1-\delta)\bar{\mu}_g(B)$, such that $\left|\frac{\mathrm{d}\bar{\mu}_B^p}{\mathrm{d}\bar{\mu}_g}(x) - 1\right| < \delta$ for all $x \in G$.*

By [CH96, Lemma 5.1], such coverings exist for any $\delta > 0$, and for Cantor boxes arbitrarily small. The proof essentially uses the covering of $M^{\mathrm{reg}}$ from Lemma 3.6.9 to build Cantor boxes, up to finite subdivision of the covering to ensure a). To get b), subdivide the boxes into smaller ones on which the Jacobian of the holonomy map between unstable manifolds is nearly 1. This argument relies on Lusin's theorem and goes through in our setting with no changes.

*Step 2. Proof that $\bar{\alpha}_i$ is vwb.* First, define $\bar{\alpha}_i$ to be the partition of $\tilde{\Omega}$ into sets of the form $\tilde{\Omega}_0 \cap (A \times [\frac{l}{2^i}, \frac{l+1}{2^i}))$, where $A \in \mathcal{M}_{-1}^1$ and $l \in \mathbb{N}$. Then $\bar{\alpha}_0 \leqslant \bar{\alpha}_1 \leqslant \bar{\alpha}_2 \leqslant \dots$ is such that $\bigvee_{i=1}^{\infty} \bigvee_{n=-\infty}^{+\infty} \phi_n \bar{\alpha}_i$ generates the whole $\sigma$-algebra on $\tilde{\Omega}$. Using Theorems 4.1 and 4.2 from [CH96], we only need to prove that each partition $\bar{\alpha}_i$ is vwB in order to prove that $(\phi_t, \bar{\mu}_g)$ is Bernoulli.

Using $\mathcal{M}_{-1}^1$ as the basis elements of $\bar{\alpha}_i$ allows us to apply the bounds (3.6.6) directly since $\partial \mathcal{M}_{-1}^1 = \mathcal{S}_1 \cup \mathcal{S}_{-1}$. We can now apply the same arguments as in [CH96, Section 6.2] with the modifications described in the second part of the proof of [BD20, Proposition 7.19]. Actually, the only place where we need to be careful is [BD20, Eq. (7.33)] because of our additional horizontal cuttings. We finish the proof by dealing with this equation. We first have to recall some notation from [BD20] first.

17. The corresponding definition in [CH96] has a third condition, but it is satisfied in our setting since the stable and unstable manifolds are one-dimensional and have bounded curvature.

Fix some $i \in \mathbb{N}$, and let $\bar{\alpha} = \bar{\alpha}_i$. Let $\varepsilon > 0$ and define $\delta = e^{-(\varepsilon/C')^{2/(1-\gamma)}}$ (recalling that $\gamma > 1$), where $C' > 0$ is the constant from (3.7.2) below.

Let $\mathfrak{B} = \{B_1, B_2, \dots, B_k\}$ be a $\delta$-regular cover of $\tilde{\Omega}$ such that the diameters of the $B_i$'s are less than $\delta$. Define the partition $\pi = \{B_0, B_1, B_2, \dots, B_k\}$, where $B_0 = \Omega \setminus \cup_{i=1}^k B_i$. For each $i \geqslant 1$, let $G_i \subset B_i$ denote the set identified in Definition 3.7.3(b). Since $(\phi_{-1}, \bar{\mu}_g)$ is K-mixing, there exists an even number $N = 2m$, such that for any integers $N_0$, $N_1$ such that $N < N_0 < N_1$, $\delta$-almost every atom $A$ of $\bigvee_{N_0-m}^{N_1-m} \phi_{-i}\bar{\alpha}$, satisfies

$$\left| \frac{\bar{\mu}_g(B|A)}{\bar{\mu}_g(B)} - 1 \right| < \delta, \quad \text{for all } B \in \pi,$$

where $\bar{\mu}_g(\cdot|A)$ is the measure $\bar{\mu}_g$ conditioned to $A$. Now let $m$, $N_0$, $N_1$ be given as above and define $\omega = \bigvee_{N_0-m}^{N_1-m} \phi_{-i}\bar{\alpha}$. In order to prove that $\bar{\alpha}$ is vwb, we need to give estimates on elements of $\omega$. To do so, we identify as in [CH96, Section 6.2] sets of *bad* atoms, whose union will have measure less than $c\varepsilon$. As in [CH96], we call these sets $\hat{F}_1, \hat{F}_2, \hat{F}_3, \hat{F}_4$. Since the estimates on the $\bar{\mu}_g$-measure of the bad sets $\hat{F}_1$ and $\hat{F}_2$ do not change, we skip their definitions. Now, define $F_3$ to be the set of all points $x \in \Omega \setminus B_0$ such that $W^s(x)$ intersects the boundary of the element $\omega(x)$ before it fully crosses the rectangle $\pi(x)$. Thus, if $x \in F_3$, there exists a subcurve of $W^s(x)$ connecting $x$ to the boundary of $(\phi_{-i}\bar{\alpha})(x)$ for some $i \in [N_0 - m, N_1 - m]$. Then since $\pi(x)$ has diameter less than $\delta$, $\phi_i(x)$ lies within a distance $C\tilde{\Lambda}^{-i}\delta$ of the boundary of $\bar{\alpha}$ – where $\tilde{C}_1$ and $\tilde{\Lambda} > 1$ come from the hyperbolicity of the billiard flow. Using the bound (3.6.6), the total measure of such points must add up to at most

$$\sum_{i=N_0-m}^{N_1-m} \left( \frac{C}{|\log(\tilde{C}_1\tilde{\Lambda}^{-i}\delta)|^{\gamma}} + C_{\bar{\alpha}}\tilde{C}_1\tilde{\Lambda}^{-i}\delta) \right) \leqslant C_1'|\log\delta|^{1-\gamma} + C_2'\delta \leqslant C'|\log\delta|^{1-\gamma}, \quad (3.7.2)$$

for some $C' > 0$. Letting $\hat{F}_3$ denote the union of atoms $A \in \omega$ such that $\bar{\mu}_g(F_3|A) > |\log\delta|^{\frac{1-\gamma}{2}}$, it follows that $\bar{\mu}_g(F_3) \leqslant C'|\log\delta|^{\frac{1-\gamma}{2}}$. This is at most $\varepsilon$ by choice of $\delta$.

The same precaution allows us to get the same bound on $\bar{\mu}_g(\hat{F}_4)$ as in [BD20].

Finally, the bad set to be avoided in the construction of the joining is $B_0 \cup (\cup_{i=1}^4 \hat{F}_i)$. Its measure is less than $c\varepsilon$ by choice of $\delta$. From this point, once the measure of the bad set is controlled, the rest of the proof in Section 6.3 of [CH96] can be repeated verbatim. This proves that $\bar{\alpha}$ is vwB. □

**Proposition 3.7.4.** *Under the assumptions of Proposition 3.7.1, the measure $\bar{\mu}_g$ is flow adapted*[18]*, that is,* $\log||D\phi_t||$ *is $\mu_g$-integrable.*

*Proof.* Let $\Omega = \{(x, y, \theta) \in Q \times \mathbb{S}^1\} \subset \mathbb{T}^3$ denote the phase space for the billiard flow $\Phi_t$ with the usual Euclidean metric denoted by $d_\Omega$. Let $\nu_g$ be the flow invariant measure obtained as the image of $\bar{\mu}_g$ by the conjugacy map between $\Omega$ and $\tilde{\Omega}$. Let

$$\mathcal{S}_0^- = \{\Phi_{-t}(z) \in \Omega \mid z \in \mathcal{S}_0 \text{ and } t \leqslant \tau(T^{-1}z)\}$$

denote the flow surface obtained by flowing $\mathcal{S}_0$ backward until its first collision under the inverse flow. Similarly, let

$$\mathcal{S}_0^+ = \{\Phi_t(z) \in \Omega \mid z \in \mathcal{S}_0 \text{ and } t \leqslant \tau(z)\}$$

18. This result is due to Mark Demers. I thank him for allowing me to use his proof.

denote the forward flow of $\mathcal{S}_0$ until its first collision. To show that the measure $\nu_t$ is flow-adapted, it suffices to show that $\int_\Omega |\log d_\Omega(x, \mathcal{S}_0^\pm)| \, d\nu_g(x) < \infty$. For then this implies that $\log \|D\Phi_t\|$ is integrable for each $t \in [-\tau_{\min}, \tau_{\min}]$ and then by subadditivity for each $t \in \mathbb{R}$.

Let $P^\pm(\cdot)$ denote the projection under the forward (backward) flow of a subset of $\Omega$ until first collision. Let $N_\varepsilon^M(\cdot)$ denote the $\varepsilon$-neighborhood of a set in $M$ in the Euclidean metric $d_M$ and let $N_\varepsilon^\Omega(\cdot)$ denote the $\varepsilon$-neighborhood of a set in $\Omega$ in the metric $d_\Omega$. It follows from [CM06, Exercise 3.15], that there exists $C > 0$ such that for any $\varepsilon > 0$,

$$P^-(N_\varepsilon^\Omega(\mathcal{S}_0^-)) \subset N_{C\varepsilon^{1/2}}^M(\mathcal{S}_1) \quad \text{and similarly} \quad P^+(N_\varepsilon^\Omega(\mathcal{S}_0^+)) \subset N_{C\varepsilon^{1/2}}^M(\mathcal{S}_{-1}) \tag{3.7.3}$$

From (3.6.6), there exist $C_g > 0$ and $\gamma > 1$ such that

$$\mu_g(N_\varepsilon^M(\mathcal{S}_{\pm 1})) \leqslant C_g |\log \varepsilon|^{-\gamma} \qquad \text{for all } \varepsilon > 0. \tag{3.7.4}$$

Putting together (3.7.3) and (3.7.4) yields

$$\nu_g(N_\varepsilon^\Omega(\mathcal{S}_0^-)) \leqslant \tau_{\max} C_g |\log C\varepsilon^{1/2}|^{-\gamma} \leq C' \tau_{\max} |\log \varepsilon|^{-\gamma} . \tag{3.7.5}$$

For $p > 1$ to be chosen below, define for $n \geqslant 1$, $A_n = N_{e^{-n^p}}^\Omega(\mathcal{S}_0^-) \setminus N_{e^{-(n+1)^p}}^\Omega(\mathcal{S}_0^-)$. If $x \in A_n$, then $|\log d_\Omega(x, \mathcal{S}_0^-)| \leqslant (n+1)^p$. Thus we estimate using (3.7.5),

$$\begin{aligned} \int_\Omega |\log d_\Omega(x, \mathcal{S}_0^-)| \, d\nu_g &\leqslant 1 + \log \operatorname{diam}(\Omega) + \sum_{n \geq 1} \int_{A_n} |\log d_\Omega(x, \mathcal{S}_0^-)| \, d\nu_g \\ &\leqslant 1 + \log \operatorname{diam}(\Omega) + \sum_{n \geq 1} (n+1)^p C' \tau_{\max} n^{-\gamma p} , \end{aligned}$$

and the series converges as long as $p > 1/(\gamma - 1)$. A similar argument shows that $\log d_\Omega(x, \mathcal{S}_0^+)$ is $\nu_g$ integrable so that $\nu_g$ is flow adapted. □

## 3.A Motivations from uniform hyperbolic dynamics

We start this note by presenting the usual method the existence of measures of maximal entropy is proved in the case of uniform hyperbolicity. First, we consider a hyperbolic transformation of a compact set, and then the case of an Anosov flow.

### 3.A.1 Hyperbolic maps

Let $X$ be a compact Riemannian manifold and let $T : X \to X$ be a $\mathcal{C}^r$ diffeomorphism. Assume that $T$ is uniformly hyperbolic, that is

$$\begin{aligned} &\exists \lambda > 1, \exists C > 0, \exists E^s, E^u \subset TX \text{ such that} \\ &(i)\, TX = E^s \oplus E^u,\, DT(E^s) \subset E^s,\, DT^{-1}(E^u) \subset E^u, \\ &(ii)\, ||D_x T^n v_s|| \leqslant C\lambda^{-n} ||v_s||, \quad \forall n \geqslant 0, \forall v_s \in E^s_x \subset T_x X, \\ &(iii)\, ||D_x T^{-n} v_u|| \leqslant C\lambda^{-n} ||v_u||, \quad \forall n \geqslant 0, \forall v_u \in E^u_x \subset T_x X. \end{aligned}$$

One fundamental theorem about hyperbolic dynamic is the Hadamard–Perron Theorem [KH95, Theorem 6.2.8] which states that there exist two unique families of $\mathcal{C}^r$ manifolds, $\{W_m^+\}_{m \in \mathbb{Z}}$ and $\{W_m^-\}_{m \in \mathbb{Z}}$, everywhere tangent respectively to $E^s$ and to $E^u$, obtained as the graph of some functions, and satisfying some stability and contraction properties. A key tool in the proof is the construction of families of stable and unstable cones.

As a consequence [KH95, Corollary 6.4.10], all such diffeomorphisms are expansive, that is

$$\exists \delta > 0,\ \forall x, y \in X,\ [\mathrm{d}(T^n(x), T^n(y)) < \delta,\ \forall n \in \mathbb{Z} \Rightarrow x = y]. \tag{3.A.1}$$

From the expansive property, it follows from [Wal82, Theorem 8.2] that the metric entropy $\mu \mapsto h_\mu(T)$ is upper semi-continuous, hence the existence of equilibrium states for every continuous potential – and in particular existence of measures of maximal entropy for the zero potential. In the proof of [Wal82, Theorem 8.2], expansiveness is only use to get the equality $h_\mu(T) = h_\mu(T, \mathcal{A})$ for partition $\mathcal{A}$ with $\mathrm{diam}(\mathcal{A}) < \delta$ (the expansivity constant of $T$) and any $T$-invariant measure $\mu$.

As proved by Bowen [Bow72a, Theorem 3.5], the expansiveness assumption of [Wal82, Theorem 8.2] can be weakened to *entropy-expansiveness* (the proof remains unchanged). This weakening will be relevant in the case of Anosov flows.

### 3.A.2 Anosov flows

Let $X$ be a compact manifold and $\phi = \{\varphi^t\} : \mathbb{R} \times X \to X$ be a smooth flow. Assume that $\phi$ is an Anosov flow, that is

$$\begin{aligned} &\exists \lambda > 1, \exists C > 0, \exists E^s,\ E^u,\ E^c \subset TX \text{ such that} \\ &(i)\ TX = E^c \oplus E^s \oplus E^u, \\ &(ii)\ D\varphi^t(E^{s/u}) = E^{s/u},\ \mathrm{dim} E^c_x = 1,\ \frac{d}{dt}\Big|_{t=0} \varphi^t(x) \in E^c_x \setminus \{0\}, \\ &(iii)\ \|D_x\varphi^t_{|E^s_x}\| \leqslant C\lambda^{-t},\ \|D_x\varphi^{-t}_{|E^u_x}\| \leqslant C\lambda^{-t},\ \forall t \geqslant 0. \end{aligned}$$

In [Bow72b, Proposition 1.6], Bowen proves that an Anosov flow is *flow expansive* (in the sense of Bowen–Walters), that is – as defined in [BW72] in the case of a fixed-point free flow,

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in X,\ \forall h \in \mathcal{C}^0(\mathbb{R}) \text{ with } h(0) = 0, \\ &[d(\varphi^t(x), \varphi^{h(t)}(y)) < \delta,\ \forall t \in \mathbb{R} \Rightarrow y \in \varphi^{]-\varepsilon,\varepsilon[}(x)]. \end{aligned} \tag{3.A.2}$$

The key ingredient of the proof is the local product structure for hyperbolic flows. From (3.A.2), it is easy to see, for $h = id$, that an Anosov flow satisfies the following weaker property

$$\begin{aligned} &\exists \varepsilon > 0,\ \exists s > 0,\ \forall x \in X, \\ &\Gamma_\varepsilon(x) := \{y \in X \mid \forall t \in \mathbb{R},\ d(\varphi^t(x), \varphi^t(y)) < \varepsilon\} \subset \varphi^{[-s,s]}(x). \end{aligned} \tag{3.A.3}$$

Bowen proved [Bow72a, Example 1.6] that (3.A.3) is a sufficient condition so that every time $\varphi^t$ of the flow is entropy-expansive. Therefore the map $\mu \in \mathcal{M}_X(\varphi^1) \mapsto h_\mu(\varphi^1)$ is upper semi-continuous, and so is its restriction to $\mathcal{M}_X(\phi) \subset \mathcal{M}_X(\varphi^1)$. Hence, Anosov flows have equilibrium states for every continuous potential, and in particular for the zero potential, measures of maximal entropy.

## 3.B Obstructions for the Billiard Flow

In the previous section, in both situations, proofs of existence of MME use some sort of expansiveness. However, the existence of a local product structure is a key ingredient in order to establish the expansivity property: it gives a scale used as the $\delta$ in (3.A.1) and the $\varepsilon$ in (3.A.3). Furthermore, the uniform contraction of stable (resp. unstable) manifolds for large positive (resp. negative) times is used, and not some estimates of their lengths in negative (resp. positive) times (such as fragmentation or growth lemmas, see for example [CM06]).

### 3.B.1 Entropy expansiveness

In Bowen's proof, the local product structure is the main tool in order to prove flow expansiveness. In the case of the billiard flow, their is no such structure. Indeed, stable and unstable manifolds exist only for Lebesgue-almost every point and there is no deterministic control of their length (hence no uniform scale for a local structure). One might argue that a billiard flow admits invariant “cone" fields [BDL18, Section 2] and construct stable and unstable curves, but then the control on the length of those curves when applying the flow is in term of expansion, not in term of contraction.

It then seems that h-expansiveness of each time $\varphi^t$ of the flow is too much to ask for. Still, one might hope that each $\varphi^t$ is *asymptotically h-expansive*, that is $h^*(\varphi^t) := \lim_{\varepsilon\to 0} h^*(\varphi^t,\varepsilon) = 0$, where $h^*(\varphi^t,\varepsilon) = \sup_{x\in X} h(\varphi^t, \overline{B}(x,\varepsilon))$. This definition was first introduced by Misiurewicz in [Mis73] where he proved that the metric entropy of an asymptotic h-expansive transformation is upper semi-continuous.

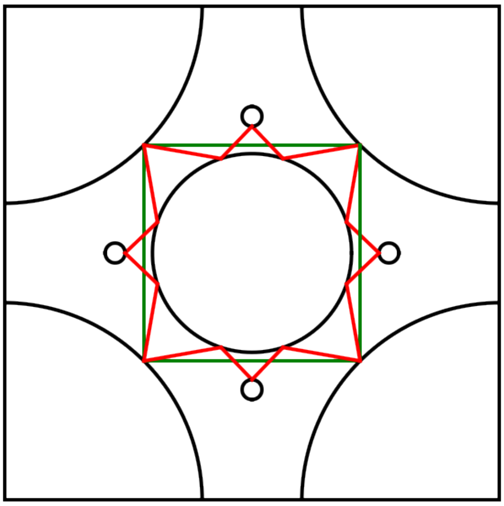
The quantity $h^*(\varphi^t)$ is usually referred to as the *topological tail entropy* of $\varphi^t$ [Dow11]. In the context of smooth dynamics, Buzzi [Buz97] has shown that if $f \in \mathcal{C}^r(M)$, then $h^*(f) \leqslant \frac{dim(M)R(f)}{r}$ for some constant $R(f)$. In particular, the metric entropy of a $\mathcal{C}^\infty$ transformation is upper semi-continuous. Clearly, this result does not apply to billiard flows.

Proving that the topological tail entropy of the billiard flow is zero is enough to prove the upper semi-continuity of the metric entropy, hence the existence of some measure of maximal entropy.

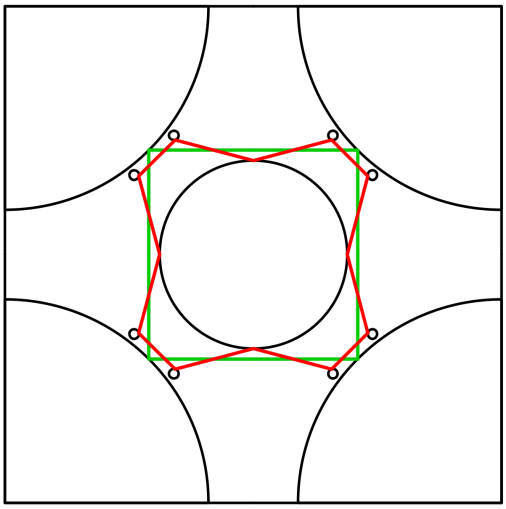
### 3.B.2 Relations with the Collision Map

In [BW72, Theorem 6], Bowen and Walters prove that the special flow constructed over a continuous transformation and under a continuous return time function, is flow expansive if and only if the base map is expansive. Since flow expansiveness is an invariant for flow under reparametrization, without loss of generality, the return time function can be chosen constant.

In [BD20], Baladi and Demers show that the collision map is expansive. However, since the return time is only piecewise continuous, it is not easy to relate the expansivity of the collision map to flow expansiveness of the billiard flow. As shown in Figure 3.2, two trajectories can be easily separated by the collision map, but they remain *close* in the phase space of the flow. We see that for a $\delta$ too large in (3.A.2) (and a natural choice of $h$), the two trajectories cannot be distinguished. What could be a good choice for $\delta$? The main problem being to find a $\delta$ independent of trajectories (it is *easier* to find a $\delta$



(a) Some common collisions



(b) Distinct collisions

**Figure 3.2** – Two examples of two periodic trajectories.

for specific trajectories, such as those ones in Figure 3.2, but the inf of those $\delta$ over all trajectories might be 0). If such $\delta$ existed, we expect it is controlled in some way by $\tau_{min}$.

For similar reasons, it appears that it is not a simple consequence of the collision map expansiveness for the flow to satisfy condition (3.A.3) (which is a weaker than *flow expansiveness*). For example, the two orbits shown in Figure 3.2 (b) are close in the phase space of the flow, but far apart in the phase space of the collision map (since the collisions they make are distinct).

# Chapter 4

## Measure of maximal entropy for finite horizon Sinai billiard flows


**Abstract**

This chapter contains the results of [BCD22]. Using results from Chapter 3 on equilibrium states for the billiard map, and bootstrapping via a "leapfrogging" method from [BD22], we construct the unique measure of maximal entropy (MME) for two-dimensional finite horizon Sinai (dispersive) billiard flows $\Phi^1$ (and show it is Bernoulli), assuming the bound $h_{\text{top}}(\Phi^1)\tau_{\min} > s_0 \log 2$, where $s_0 \in (0,1)$ quantifies the recurrence to singularities. This bound holds in many examples (it is expected to hold generically).


## 4.1 Introduction and Main Result

First, recall the set-up and some notations: A Sinai billiard table $Q$ on the two-torus $\mathbb{T}^2$ is a set $Q = \mathbb{T}^2 \setminus \cup_i \mathcal{O}_i$, for finitely many pairwise disjoint closed domains $\mathcal{O}_i$ with $C^3$ boundaries having strictly positive curvature $\mathcal{K}$. The billiard flow $\Phi^t$, $t \in \mathbb{R}$, is the motion of a point particle traveling in $Q$ at unit speed and undergoing specular reflections [1] at the boundary of the scatterers $\mathcal{O}_i$. The associated billiard map $T : M \to M$, on the compact metric set $M = \partial Q \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, is the first collision map on the boundary of $Q$. Grazing collisions cause discontinuities in the map $T$, but the flow is continuous. However, it is not obvious that the flow satisfies a condition (such as asymptotic $h$-expansiveness) sufficient for the upper-semi continuity of the Kolmogorov entropy (see Chapter 3 Appendices A and B). There thus does not appear to exist any quick way to prove that the billiard flow admits a measure of maximal entropy.

To state our main results, Theorem 4.1.4 and [2] Corollary 4.1.5, we introduce some basic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1. At a tangential collision, the reflection does not change the direction of the particle.
2. The condition (4.1.4) there is discussed in Lemma 4.1.3.

notations. Recall that for $x \in M$, $\tau(x)$ denotes the flow time from $x$ to $T(x)$, and set

$$\tau_{\min} = \inf \tau > 0\,, \ \tau_{\max} = \sup \tau\,, \ \Lambda = 1 + 2\tau_{\min} \inf \mathcal{K}\,.$$

Throughout, we assume finite horizon, that is: there are no trajectories making only tangential collisions. Finite horizon implies $\tau_{\max} < \infty$.

The topological entropy $h_{\text{top}}(\Phi^1)$ of the continuous map $\Phi^1$ is the supremum of the Kolmogorov entropies $h_\nu(\Phi^1)$ of the ergodic $\Phi^1$-invariant probability measures. Set

$$P(t) = \sup_{\mu: T\text{-invariant ergodic probability measure}} \{h_\mu(T) - t\int \tau d\mu\}\,, \ t \geqslant 0\,.$$

The real number $P(t)$ is called the pressure of the potential $-t\tau$ and a probability measure $\mu_t$ realising $P(t)$ is called an equilibrium measure for $-t\tau$.

Viewing $\Phi$ as the suspension of $T$ under $\tau$, Abramov's formula says that any ergodic probability measure $\nu$ invariant under the time-one map $\Phi^1$ satisfies

$$\nu = \frac{\mu}{\int \tau d\mu} \otimes Leb\,, \tag{4.1.1}$$

where $\mu$ is an ergodic $T$-invariant probability measure, and, in addition,

$$h_\nu(\Phi^1) = \frac{h_\mu(T)}{\int \tau d\mu}\,. \tag{4.1.2}$$

Using the growth rate of $\mathcal{M}_0^n$, set

$$h_* = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n$$

(existence of the limit is easy [BD20]). Then, for fixed $\varphi < \pi/2$ close to $\pi/2$ and large $n \in \mathbb{N}$, we can define $s_0(\varphi, n) \in (0, 1]$ to be the smallest number such that any orbit of length equal to $n$ has at most $s_0 n$ collisions whose angles with the normal are larger than $\varphi$ in absolute value. Now, [BD20] proves that if

$$h_* > s_0 \log 2 \tag{4.1.3}$$

then $P(0) = h_*$, and there is a unique equilibrium measure $\mu_* = \mu_0$ for $t = 0$, which is the unique measure of maximal entropy (MME) of $T$. There are many billiards [BD20, §2.4] satisfying (4.1.3), and in fact we do not know any billiard which violates it. (Note also that Demers and Korepanov showed [DK22] that a conjecture of Bálint and Tóth, if true, implies that, generically, one can choose $\varphi$ and $n$ to make $s_0$ arbitrarily small.)

Using Abramov's formula, recall from Chapter 3:

**Proposition 4.1.1** (Lemma 3.2.5, Corollary 3.2.6)**.** *The real number $t = h_{top}(\Phi^1) > 0$ is the unique $t$ such that $P(t) = 0$. In addition, the set of equilibrium measures of $T$ for $-h_{top}(\Phi^1)\tau$ is in bijection with the set of MMEs of the flow via (4.1.1).*

Denote $S_n\tau := \sum_{k=0}^{n-1} \tau \circ T^k$. We recall that it follows from Theorems 3.1.2 and 3.2.1 that, in the special case of potentials of the form $-t\tau$

**Theorem 4.1.2** (Theorem 3.1.2, Theorem 3.2.1)**.** *(a) The following*[3] *limits exist:*

$$P_*(t) = \lim_{n\to\infty} \frac{1}{n} \log Q_n(t)\,, \;\; \textit{with } Q_n(t) = \sum_{A\in\mathcal{M}_0^n} |e^{-tS_n\tau}|_{C^0(A)}\,, \; \forall t \geqslant 0\,.$$

*Moreover, $P_*(t) > P_*(s) \geqslant P(s)$ for all $0 \leqslant t < s$, and*[4] *$t \mapsto P_*(t)$ is convex.*
*(b) If $t \geqslant 0$ is such that*

$$P_*(t) + t\tau_{\min} > s_0 \log 2\,, \tag{4.1.4}$$

*and*

$$\log \Lambda > t(\tau_{\max} - \tau_{\min})\,, \tag{4.1.5}$$

*then there is a unique equilibrium measure $\mu_t$ for $-t\tau$. This measure charges all open sets, is Bernoulli, and $P_*(t) = P(t)$. Finally, $\mu_t$ is $T$-adapted,*[5] *that is*

$$\int |\log d(x, \mathcal{S}_{\pm 1})|\, d\mu_t < \infty\,. \tag{4.1.6}$$

In view of Proposition 4.1.1 and Theorem 4.1.2, to establish existence and uniqueness of the MME of the finite horizon flow $\Phi$, it *suffices* to check (4.1.4) and (4.1.5) for $t = h_{\text{top}}(\Phi^1) > 0$. We next discuss these conditions. The first one is very mild:

**Lemma 4.1.3.** *The bound (4.1.4) holds at $t = h_{top}(\Phi^1)$ as soon as*

$$h_{top}(\Phi^1)\tau_{\min} > s_0 \log 2\,. \tag{4.1.7}$$

*The bound (4.1.7) holds as soon as*

$$h_* \frac{\tau_{\min}}{\tau_{\max}} > s_0 \log 2\,. \tag{4.1.8}$$

*If (4.1.4) holds for some $t' \geqslant 0$ then it holds for all $t \in [0, t']$.*

It is not hard to find (see Remark 3.5.6) billiards satisfying (4.1.7).

*Proof.* The first claim follows from Proposition 4.1.1 and the bound $P_*(t) \geqslant P(t)$ for all $t \geqslant 0$. The second claim holds because (4.1.2) implies $h_{\text{top}}(\Phi^1) \geqslant \frac{h_*}{\int \tau d\mu_*} \geqslant \frac{h_*}{\tau_{\max}}$. Finally, the first claim of Lemma 4.3.3 below implies that $t \mapsto P_*(t) + t\tau_{\min}$ is nonincreasing. □

The second condition (4.1.5) will require more efforts. Obviously, for any finite horizon billiard, there exists $\tilde{t} > 0$ such that (4.1.5) holds for all $t \in [0, \tilde{t}]$. However, we do[6] not know any billiard such that (4.1.5) holds for $t = h_{\text{top}}(\Phi^1)$ (that is, $\log \Lambda > h_{\text{top}}(\Phi^1)(\tau_{\max} - \tau_{\min})$). Fortunately, it turns out that (4.1.5) is not *necessary:* Assuming only finite horizon and (4.1.4) at $t = h_{\text{top}}(\Phi^1)$, we will extend the conclusion of Theorem 4.1.2 to $t = h_{\text{top}}(\Phi^1)$ by adapting the bootstrapping argument in [BD22, Lemma 3.10] (used there to cross the value $x = 1$ at which the pressure for $-x \log J^u T$ vanishes). This is our main result:

3. By [BD20] we always have $P_*(0) = h_* \geqslant P(0)$.
4. The fact that $P_*(t)$ is strictly decreasing is immediate, see (4.3.5). Convexity follows from the Hölder inequality as in [BD22, Prop 2.6].
5. To establish (4.1.6), Carrand shows that the $\mu_t$ measure of the $\epsilon$-neighbourhood of $\mathcal{S}_{\pm 1}$ is bounded by $C_t|\log \epsilon|^\gamma$ for $\gamma > 1$ and $C_t < \infty$.
6. Note that (4.1.2) implies $h_{\text{top}}(\Phi^1)(\tau_{\max} - \tau_{\min}) \leqslant h_*(\tau_{\max}/\tau_{\min} - 1)$.

**Theorem 4.1.4.** *Let $T$ be a finite horizon Sinai billiard map such that (4.1.4) holds at $t = h_{top}(\Phi^1)$. Then for all $t \in [0, h_{top}(\Phi^1)]$, we have $P_*(t) = P(t)$, and there exists a unique $T$-invariant probability measure $\mu_t$ realising $P(t)$. This measure charges all nonempty open sets, is Bernoulli and $T$-adapted.*

Our proof furnishes $t_\infty \geqslant h_{\mathrm{top}}(\Phi^1)$ such that the key Small Singular Pressure properties (see Definitions 3.3.2 and 3.3.5) hold for $g = -t\tau$ for all $t \in [0, t_\infty]$. If $t_\infty > h_{\mathrm{top}}(\Phi^1)$ and if (4.1.4) holds for some $t_2 \in (h_{\mathrm{top}}(\Phi^1), t_\infty]$, then the conclusion of Theorem 4.1.4 holds for all $t \in [0, t_2]$.

Theorem 4.1.2 and Proposition 4.1.1, combined with Theorem 4.1.4 and the proof of Propositions 3.7.1 and 3.7.2 for Bernoullicity of the flow, give:

**Corollary 4.1.5.** *Let $T$ be a finite horizon Sinai billiard map such that (4.1.4) holds at $t = h_{top}(\Phi^1)$. Then*

$$\nu_* := \frac{\mu_{h_{top}(\Phi^1)}}{\int \tau \, d\mu_{h_{top}(\Phi^1)}} \otimes Leb$$

*is the unique measure of maximal entropy of the billiard flow. This measure is Bernoulli, it charges all nonempty open sets, and it is flow adapted, that is*[7]

$$\int_\Omega |\log d_\Omega(x, \mathcal{S}_0^\pm)| \, d\nu_* < \infty \,, \quad \Omega = Q \times \mathbb{S}^1 \,, \tag{4.1.9}$$

*where $d_\Omega$ is the Euclidean metric, $\mathcal{S}_0^- = \{\Phi_{-s}(z) : z \in \mathcal{S}_0 \,,\, s \leqslant \tau(T^{-1}z)\}$, and $\mathcal{S}_0^+ = \{\Phi_s(z) : z \in \mathcal{S}_0 \,,\, s \leqslant \tau(z)\}$.*

Contrary to [BD22], homogeneity layers are not used for our potentials $-t\tau$. They are not needed because $\tau$ is piecewise Hölder and thus $e^\tau$ satisfies piecewise bounded distortion. The results from Chapter 3 that we build upon are based on bounds for transfer operators acting on Banach spaces of distributions defined with the logarithmic modulus of continuity of [BD20]. We could not find a Banach norm giving a spectral gap (there is no analogue of [BD22, Lemmas 3.3 and 3.4] for $\varsigma \neq 0$, see Lemma 3.3.1 for $\gamma \neq 0$ where $(\log |W|/\log |W_i|)^\gamma$ replaces $(|W_i|/|W|)^\varsigma$). We thus do not have exponential mixing for $(T, \mu_{h_{\mathrm{top}}(\Phi^1)})$. (Even if we had, it would not immediately imply exponential mixing for $(\Phi^1, \nu_*)$.)

This chapter is organised as follows: Section 4.2 is devoted to recalling notation from [BD20] and to two basic lemmas on cone stable curves iterated by the billiard map. Section 4.3 is the core of the paper: In §4.3.1, after recalling in the present context the Small Singular Pressure (SSP) conditions (4.3.1), (4.3.2), and (4.3.3) and the main result of Chapter 3 (Theorem 4.3.1), we reduce Theorem 4.1.4 to showing SSP for some $t \geqslant h_{\mathrm{top}}(\Phi^1)$ (Lemma 4.3.2). Then we set up the bootstrap mechanism, by introducing in (4.3.4) the supremum $t_\infty > 0$ of parameters satisfying SSP (this is the new idea). Lemma 4.3.3 embodies our version of the first ingredient of the bootstrap from [BD22, Definition 3.9] ("pressure gap"), constructing a "pivot" $t_* < t_\infty$ and its associated parameter $s_*(t_*) > t_\infty$. The key lemmas inspired by the second ingredient of bootstrapping [BD22, Lemmas 3.10–3.11] ("leapfrogging across $t_*$ via the Hölder inequality"), are stated and proved in §4.3.2. Finally, Lemma 4.3.2 (and thus Theorem 4.1.4) is proved in §4.3.3: We assume for a

7. Note that (4.1.9) implies that $\log \|D\Phi_t\|$ is integrable for each $t \in [-\tau_{\min}, \tau_{\min}]$ so that, by subadditivity, it is integrable for each $t \in \mathbb{R}$.

contradiction that $t_\infty < h_{\mathrm{top}}(\Phi^1)$. Since $t_* < t_\infty$, this implies, by results from Chapter 3 recalled in Proposition 4.1.1 and Theorem 4.1.2(a), that the pressure of $t_*$ is positive. Then, we exploit this positivity in order to pass over the pivot $t_*$ via the key lemmas from §4.3.2, obtaining the desired contradiction.

Observe that the results contained in Chapter 3 apply to families more general than $g_t = -t\tau$, the results of the present chapter extend to suitable one parameter-families $g_t$ of piecewise Hölder potentials. We abstain from spelling out the details.

## 4.2 Notations. $n$-step Expansion. Growth Lemma

We recall here some facts about hyperbolicity and complexity of finite horizon Sinai billiards. There exist continuous families of stable and unstable cones, $\mathcal{C}^s$ and $\mathcal{C}^u$, which can be taken constant in $M$, and a constant $C_1 \in (0,1)$ such that,

$$\|DT^n(x)v\| \geqslant C_1\Lambda^n\|v\|\,,\ \forall v \in \mathcal{C}^u\,,\quad \|DT^{-n}(x)v\| \geqslant C_1\Lambda^n\|v\|\,,\ \forall v \in \mathcal{C}^s\,, \tag{4.2.1}$$

where, as before, $\Lambda = 1 + 2\tau_{\min}\mathcal{K}_{\min}$ is the minimum hyperbolicity constant.

A fundamental fact about this class of billiards is the linear bound on the growth in complexity due to Bunimovich [Che01, Lemma 5.2],

There exists $K \geqslant 1$ such that for all $n \geqslant 0$, the number of curves in $\mathcal{S}_{\pm n}$ that intersect at a single point is at most $Kn$. (4.2.2)

The parameter $\gamma > 1$ defining the Banach space norms in Section 3.4.2 is chosen so that $h_* > s_0\gamma\log 2$, which is possible due to (4.1.3). Next, choosing $m$ so large that,

$$\tfrac{1}{m}\log(Km+1) < h_* - s_0\gamma\log 2\,,$$

we take $\delta_0 = \delta_0(m) \in (0, 1/C_1)$ so that any stable curve of length at most $\delta_0$ can be cut by $\mathcal{S}_{-\ell}$ into at most $K\ell + 1$ connected components for all $0 \leqslant \ell \leqslant 2m$.

Let $\widehat{\mathcal{W}}^s$ be, as in [BD20, §5], the set of (cone-stable) curves whose tangent vectors lie in the stable cone for $T$, with length at most $\delta_0$ and curvature bounded above by a constant $C_{\mathcal{K}}$ depending only on the table (homogeneity layers are not used). The constant $C_{\mathcal{K}}$ is chosen large enough that $T^{-1}\widehat{\mathcal{W}}^s \subset \widehat{\mathcal{W}}^s$, up to subdivision of curves. For $n \geqslant 1$, $\delta \in (0, \delta_0]$, and $W \in \widehat{\mathcal{W}}^s$, let $\mathcal{G}_n^\delta(W)$, $L_n^\delta(W)$, $S_n^\delta(W)$, and $\mathcal{I}_n^\delta(W)$ be as in [BD20, §5]: Set $\mathcal{G}_0^\delta(W) = W$ and define $\mathcal{G}_n^\delta(W)$ for $n \geqslant 1$ to be the set of smooth components of $T^{-1}W'$ for $W' \in \mathcal{G}_{n-1}^\delta(W)$, with elements longer than $\delta$ subdivided to have length between $\delta/2$ and $\delta$. More precisely, if a smooth component $U$ has length $\ell\delta + \rho$ with $\ell \geqslant 1$ and $0 \leqslant \rho < \delta$, we decompose $U$ into:

- either $\ell \geqslant 2$ pieces of length $\delta$, if $\rho = 0$,
- or $\ell \geqslant 1$ piece(s) of length $\delta$ and one piece of length $\rho$, placed at one of the edges of $U$, if $\rho \geq \delta/2$,
- or $\ell - 1 \geqslant 0$ piece(s) of length $\delta$, one piece of length $\delta/2$ (at one tip) and one piece of length $\rho + \delta/2$ (at the other tip), if $\rho \in (0, \delta/2)$.

Let $L_n^\delta(W)$ denote the set of curves in $\mathcal{G}_n^\delta(W)$ that have length at least $\delta/3$ and let $S_n^\delta(W) = \mathcal{G}_n^\delta(W) \setminus L_n^\delta(W)$. For $0 \leqslant k < n$, we say that $U \in \mathcal{G}_k^\delta(W)$ is an ancestor of $V \in \mathcal{G}_n^\delta(W)$ if $T^{n-k}V \subseteq U$, and we define $\mathcal{I}_n^\delta(W)$ to be those curves in $\mathcal{G}_n^\delta(W)$ that have no ancestors of length at least $\delta/3$ (aside from perhaps $W$ itself).

Finally, let $\delta_1 < \delta_0$ and $n_1 \geqslant m$ be chosen so that [BD20, eq. (5.6)] holds: For any stable curve $W$ with $|W| \geqslant \delta_1/3$ and $n \geqslant n_1$,

$$\# L_n^{\delta_1}(W) \geqslant \tfrac{2}{3} \# \mathcal{G}_n^{\delta_1}(W) .$$

Up to replacing $\delta_1$ by a smaller constant, we may and shall only consider values of $\delta$ of the form $\delta_0/2^N$ for $N \geqslant 0$. By induction on $N$, selecting the short tips in a compatible way when dividing $\delta$ by two, we require that [8] for all $W \in \widehat{\mathcal{W}}^s$,

$$\forall n \geqslant 1\,,\ \text{if } \delta'' < \delta' \text{ then } \forall\, U'' \in L_n^{\delta''}(W)\,,\ \exists!\, U' \in \mathcal{G}_n^{\delta'}(W) \text{ with } U'' \subset U'\,, \tag{4.2.3}$$

For $t \geqslant 0$, we introduce the following shorthand notation,

$$S_n^\delta(W,t) := \sum_{W_i \in S_n^\delta(W)} |e^{-tS_n\tau}|_{C^0(W_i)}\,,\ \mathcal{G}_n^\delta(W,t) := \sum_{W_i \in \mathcal{G}_n^\delta(W)} |e^{-tS_n\tau}|_{C^0(W_i)}\,,$$

and

$$L_n^\delta(W,t) := \mathcal{G}_n^\delta(W,t) - S_n^\delta(W,t)\,,\ \mathcal{I}_n^\delta(W,t) := \sum_{W_i \in \mathcal{I}_n^\delta(W)} |e^{-tS_n\tau}|_{C^0(W_i)}\,.$$

The lemma below replaces the usual one-step expansion (see [BD22, Lemma 3.1]):

**Lemma 4.2.1** ($n$-Step Expansion). *For any $t_0 > 0$ and $\theta_0 \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$ there exist a finite $n_0(t_0,\theta_0) \geqslant 2$ and $\bar\delta_0 = \frac{\delta_0}{2^N} > 0$ such that*

$$S_{n_0}^{\bar\delta_0}(W,t) \leqslant \mathcal{G}_{n_0}^{\delta_0}(W,t) < \theta_0^{n_0 t}\,,\quad \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \leqslant \bar\delta_0\,,\ \forall t \geqslant t_0\,. \tag{4.2.4}$$

See also Lemma 3.3.1(a).

*Proof.* Clearly, $\sup -t\tau \leqslant -t\tau_{\min} < 0$ if $t > 0$. For any $n_0 \geqslant 1$, there exists $\bar\delta_0(n_0) = \frac{\delta_0}{2^N}$ such that any $W \in \widehat{\mathcal{W}}^s$ with $|W| < \bar\delta_0$ is such that $T^{-n_0}(W)$ has at most $(Kn_0+1)$ connected components [Che01, Lemma 5.2]. In addition using [CM06, Ex. 4.50] as in [BD20, Proof of Lemma 5.1], we have $|T^{-j}W| \leqslant C'|W|^{2^{-s_0 j}}$ for a uniform $C' > 0$ and all $j \geqslant 1$ (see also Lemma 3.3.1). Up to taking smaller $\bar\delta_0$, depending on $\delta_0$ (and $n_0$), we can assume that $|T^{-j}W| \leqslant \delta_0$ for all $0 \leqslant j \leqslant n_0$. Then, for $|W| \leqslant \bar\delta_0$, there can be no additional subdivisions of $T^{-n_0}(W)$ due to pieces growing longer than $\delta_0$, so that

$$\mathcal{G}_{n_0}^{\delta_0}(W,t) \leqslant (Kn_0+1)e^{-tn_0\tau_{\min}}\,. \tag{4.2.5}$$

The same bound applies to $S_{n_0}^{\bar\delta_0}(W,t)$, since any element of $S_{n_0}^{\bar\delta_0}(W)$ must be created by a genuine cut by a singularity, not an additional subdivision due to pieces growing longer than $\bar\delta_0$. For any fixed $t_0 > 0$ and $\theta_0 \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$, we can find $n_0 = n_0(t_0,\theta_0) \geqslant 2$ such that $(Kn_0+1)^{1/n_0} \leqslant \theta_0^{t_0} e^{\tau_{\min} t_0}$. Since $\theta_0^{t_0} e^{\tau_{\min} t_0} \leqslant \theta_0^{t} e^{\tau_{\min} t}$ for all $t \geqslant t_0$, it follows that (4.2.4) holds for $\bar\delta_0 = \bar\delta_0(n_0,\delta_0)$. □

8. We use this in the proof of Lemma 4.3.7 below. An alternative way to guarantee (4.2.3) for a fixed length scale $\delta'$ is to define $\mathcal{G}_n^{\delta'}(W)$ as usual and treat it as the canonical partition of $T^{-n}W$. Then for any $\delta'' < \delta'/2$ one can define $\mathcal{G}_n^{\delta''}(W)$ as a refinement of $\mathcal{G}_n^{\delta'}(W)$, guaranteeing (4.2.3). This is done implicitly in the proof of [BD22, Lemma 3.11] and could be applied in our Lemma 4.3.7 below by taking $\delta' = \delta_{t_*}$ of that lemma. We do not adopt this approach since the canonical scale would not be chosen until nearly the end of our proof.

Lemma 4.2.1 implies the following analogue[9] of [BD22, Lemmas 3.3–3.4, $\zeta = 0$]:

**Lemma 4.2.2** (Growth Lemma). *Fix $\theta_0 \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$ and $t_0 > 0$. Suppose $\delta \leqslant \delta_0$ and $m_1(\delta) \geqslant n_0(t_0, \theta_0)$ are such that any $W \in \widehat{\mathcal{W}}^s$ with $|W| \leqslant \delta$ has the property that $W \setminus \mathcal{S}_{-j}$ comprises at most $Kj + 1$ connected components for all $1 \leqslant j \leqslant 2m_1$. Then for any $t \geqslant t_0$ and each $W \in \widehat{\mathcal{W}}^s$ with $|W| \leqslant \delta$, we have*

$$\mathcal{I}_n^\delta(W, t) \leqslant \theta_0^{nt} \, , \, \forall n \geqslant m_1 \, , \tag{4.2.6}$$

$$\mathcal{I}_n^\delta(W, t) \leqslant K m_1 \theta_0^{nt} \, , \, \forall n < m_1 \, , \tag{4.2.7}$$

*and*

$$\mathcal{G}_n^\delta(W, t) \leqslant \frac{4}{C_1 \delta} Q_n(t) \, , \forall n \geqslant 1 \, . \tag{4.2.8}$$

*Proof.* Let $n_0(t_0, \theta_0)$ and $\bar{\delta}_0(n_0, \delta_0)$ be given by Lemma 4.2.1. By choice of $n_0$, if $\varepsilon = \tau_{\min} + \log \theta_0 > 0$, then $(Kn_0 + 1)^{1/n_0} \leqslant e^{\varepsilon t_0}$. Remark that $(Kn + 1)^{1/n}$ decreases to 1 for $n \geqslant 2$ since $K \geq 1$. Thus $(Kn + 1)^{1/n} \leqslant e^{\varepsilon t_0}$ for all $n \geqslant n_0$. With this observation, for $\delta$ and $m_1$ as in the statement of the lemma, the bound (4.2.6) can be proved by induction on $n$ (just like [BD22, Lemma 3.3] for $\zeta = 0$), writing $n = qm_1 + \ell$, with $q \geqslant 1$ and $0 \leqslant \ell < m_1$, using $q - 1$ times the bound (4.2.5) with $m_1$ iterates in place of $n_0$, and using it one last time with $m_1 + \ell$ iterates, since elements of $\mathcal{I}_n^\delta(W)$ have been short at each intermediate step.

For $n < m_1$, the bound (4.2.7) follows from the relation between $\delta$ and $m_1$.

Finally, to show (4.2.8), first note that, since each $W_i \in \mathcal{G}_n^\delta(W)$ is contained in a single element of $\mathcal{M}_0^n$, and since $|T^{-n}V| \geqslant C_1 \Lambda^n |V|$ for any stable curve $|V|$ (due to (4.2.1)), there can be at most $2/(C_1\delta) + 2$ elements of $\mathcal{G}_n^\delta(W)$ in one element of $\mathcal{M}_0^n$. Note also that $|e^{-tS_n\tau}|_{C^0(W_i)} \leqslant |e^{-tS_n\tau}|_{C^0(A)}$ whenever $W_i \subset A \in \mathcal{M}_0^n$. This gives the required bound since $C_1\delta < 1$. □

## 4.3 Bootstrapping

### 4.3.1 Preparations: Small Singular Pressure. Two Bounds from Chapter 3

Recall the SSP.1 condition in the context of the family of potentials $-t\tau$: We say that Small Singular Pressure #1 (SSP.1) holds at $t \geqslant 0$ for $\varepsilon \in (0, 1/4]$ if

$$\begin{aligned} &\text{there exist } \delta_t = \delta(\varepsilon) = \frac{\delta_0}{2^{N_t}} \in (0, \delta_1] \text{ and a finite } n_t = n_t(\varepsilon) \geqslant n_1 \\ &\text{such that } \frac{S_n^{\delta_t}(W, t)}{\mathcal{G}_n^{\delta_t}(W, t)} \leqslant \varepsilon \, , \, \forall n \geqslant n_t \, , \, \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geqslant \delta_t/3 \, , \end{aligned} \tag{4.3.1}$$

and, in addition,

$$\sum_{n \geqslant n_t} \sup_{\substack{W \in \widehat{\mathcal{W}}^s \\ |W| \geqslant \delta_t/3}} \frac{e^{-nt\tau_{\min}}}{L_n^{\delta_t}(W, t)} < \infty \tag{4.3.2}$$

9. See Lemma 3.3.1(b) for the replacement for [BD22, Lemmas 3.3–3.4, $\zeta \neq 0$], using a logarithmic weight with $\gamma > 0$ as in [BD20].

together with its "time-reversal," obtained by replacing $T$ with its inverse $T^{-1}$, $\widehat{\mathcal{W}}^s$ by $\widehat{\mathcal{W}}^u$, and replacing $\tau$ with $\tau \circ T^{-1}$ (that is, replacing $S_n\tau$ with $\sum_{i=1}^n \tau \circ T^{-i} = (S_n\tau) \circ T^{-n}$), both hold.

Now, recall the SSP.2 condition in the context of the family of potentials $-t\tau$: Assume that (4.3.1) and (4.3.2) hold at $t \geqslant 0$ for $\varepsilon \leqslant 1/4$, $\delta_t$, and $n_t$. Then we say that Small Singular Pressure #2 (SSP.2) holds at $t$ for $\varepsilon$ if[10]

$$\text{for any } W \in \widehat{\mathcal{W}}^s \text{ there exists } n_t^*(|W|, \delta_t, \varepsilon) \in [n_t, \infty) \text{ such that} \qquad (4.3.3)$$
$$\frac{S_n^{\delta_t}(W,t)}{\mathcal{G}_n^{\delta_t}(W,t)} \leqslant 2\varepsilon\,,\ \forall n \geqslant n_t^*(|W|, \delta_t, \varepsilon)\,,$$

together with its time-reversal (in the sense defined above) both hold.

Note that the time-reversal of conditions (4.3.1), (4.3.2), and (4.3.3) involve stable curves for $T^{-1}$, that is, unstable curves for $T$. In view of the time reversibility of the billiard dynamics (see [CM06, Sect. 2.14] for the precise involution $\iota$), since $\tau \circ T^{-1} = \tau \circ \iota$, and $\tau \circ \iota$ is precisely the free flight time under $T^{-1}$,the conditions for $T$ and $\tau$ are equivalent[11] with those for $T^{-1} = \iota T \iota$ and $\tau \circ T^{-1} = \tau \circ \iota$.

To establish Theorem 4.1.2, recall that Theorem 3.1.2 is[12] a consequence of SSP:

**Proposition 4.3.1** (Theorem 3.1.2). *Assume*[13] (4.1.4) *and that SSP.1 and SSP.2 hold*[14] *at $t > 0$ for $\varepsilon = 1/4$. Then there is a unique equilibrium measure $\mu_t$ for $-t\tau$, this measure is $T$-adapted, charges nonempty open sets, and is Bernoulli. In addition, $P_*(t) = P(t)$.*

Therefore, to show Theorem 4.1.4 it suffices to prove the following lemma:

**Lemma 4.3.2.** *There exists $t_2 \geqslant h_{top}(\Phi^1)$ such that* (4.3.1)*,* (4.3.2)*, and* (4.3.3) *hold at all $t \in [0, t_2]$ for $\varepsilon = 1/4$.*

Setting

$$t_C = \frac{\log \Lambda}{\tau_{\max} - \tau_{\min}} > 0\,,$$

Lemmas 3.3.3, 3.3.4 and Corollary 3.3.6 give that, for any fixed $\varepsilon \in (0, 1/4]$, each $t \in [0, t_C]$ satisfies SSP (that is, (4.3.1), (4.3.2), and (4.3.3)) for $\delta_t(\varepsilon) > 0$, $n_t(\varepsilon) < \infty$, and $C_t < \infty$.

The starting point of our bootstrap argument is the following definition

$$t_\infty := \sup\{t' \geqslant 0 \text{ such that } (4.3.1), (4.3.2), \text{ and } (4.3.3) \text{ hold for all } 0 \leqslant t \leqslant t'\}\,. \qquad (4.3.4)$$

We already know that $t_\infty \geqslant t_C > 0$. If $P(t_\infty) < 0$, then $t_\infty > h_{\text{top}}(\Phi^1)$, and we have shown Lemma 4.3.2. Otherwise, Lemma 4.3.7 below will establish that any $0 \leqslant t < s_*$ satisfies SSP.1 and SSP.2 (that is, (4.3.1), (4.3.2), and (4.3.3)) where $s_* > t_\infty$ is constructed in the next lemma (inspired by [BD22, Definition 3.9]).

10. In the analogous condition of [BD20, Cor 5.3], there exists a uniform $C_t$ such that $n_t^*(|W|, \delta_t, \varepsilon) = C_t n_t \frac{|\log(|W|/\delta_t)|}{|\log \varepsilon|}$.
11. This equivalence does not always hold in Chapter 3 where $-t\tau$ is replaced by a more general $g$.
12. In particular, it is shown that (4.3.1) and (4.3.2) imply the analogues Propositions 3.3.7 and 3.3.10 of [BD22, Prop. 3.14 and 3.15] for the Banach norm of [BD20]. As opposed to [BD22], there is no spectral gap in Chapter 3.
13. See also Lemma 4.1.3.
14. SSP.1 suffices to construct the invariant measure $\mu_t$ and check it is $T$-adapted. SSP.2 is used to show ergodicity, which gives that $\mu_t$ is an equilibrium state for $-t\tau$, as well as the other claims.

**Lemma 4.3.3** (Pressure gap: Constructing the "pivot" $t_*$)**.** *For all $t > 0$, the following limit exists and belongs to $[-\tau_{\max}, -\tau_{\min}]$:*

$$P'_-(t) := \lim_{s \uparrow t} \frac{P_*(t) - P_*(s)}{t - s}\,.$$

*In addition, for any $\theta_0 \in (e^{-\tau_{\min}}, e^{-\tau_{\min}/2})$, defining*

$$s_*(t) := \frac{t|P'_-(t)|}{|P'_-(t)| + (\log \theta_0)/2}\,,\ t \in (0, t_\infty)\,,$$

*there exists $t_* \in (0, t_\infty)$ such that $s_* := s_*(t_*) > t_\infty$.*

*Remark* 4.3.4. The parameter $s_*(t) > t$ is defined so that

$$\theta_0^{s_*(t)/2} e^{|P'_-(t)|(s_*(t)-t)} = 1\,.$$

The reason for this will become clear in the proof of Lemma 4.3.7.

*Proof.* Existence of the limit follows from the convexity of $P_*(t)$ which implies that left (and right) derivatives exist at every $t > 0$. Next, if $0 < s < t$, we have

$$\sum_{A \in \mathcal{M}_0^n} |e^{-tS_n\tau}|_{C^0(A)} \leqslant |e^{n(s-t)\tau_{\min}}| \sum_{A \in \mathcal{M}_0^n} |e^{-sS_n\tau}|_{C^0(A)}\,,\ \forall n \geqslant 1\,, \tag{4.3.5}$$

which implies $P'_-(t) \leqslant -\tau_{\min}$. A similar computation gives $P'_-(t) \geqslant -\tau_{\max}$.

Next, to construct $t_*$, we first check that

$$s_*(t) > t \cdot (1 + \frac{\tau_{\min}}{4\tau_{\max}})\,,\ \forall t \in (0, t_\infty)\,. \tag{4.3.6}$$

Indeed, since

$$\frac{1}{1 - \frac{|\log \theta_0|}{2|P'_-(t)|}} \geqslant 1 + \frac{|\log \theta_0|}{2|P'_-(t)|}\,,$$

the bound (4.3.6) follows from Remark 4.3.4 and the fact that $|P'_-(t)| \leqslant \tau_{\max}$ implies

$$\frac{|\log \theta_0|}{2|P'_-(t)|} \in \Big[\frac{\tau_{\min}}{4\tau_{\max}}, 1)\,.$$

Then, taking $t_* = t_\infty - \upsilon$ for $\upsilon \in (0, t_\infty)$, it suffices to pick $\upsilon > 0$ such that

$$(1 + \frac{\tau_{\min}}{4\tau_{\max}})(t_\infty - \upsilon) > t_\infty\,.$$

Since $t_\infty \geqslant t_C = \log \Lambda / (\tau_{\max} - \tau_{\min})$, the above bound holds as soon as

$$\upsilon < \log \Lambda \cdot (\tau_{\max} - \tau_{\min})^{-1} \cdot (1 + 4\frac{\tau_{\max}}{\tau_{\min}})^{-1}\,.$$

□

We record for further use two key bounds due to Carrand. Assume that (4.3.1) (4.3.2) hold for $t$, then by Proposition 3.3.7 there exists $c_{0,t} > 0$ such that

$$\mathcal{G}_n^{\delta_t}(W, t) \geqslant c_{0,t} e^{nP_*(t)}\,,\ \forall n \geqslant 1\,, \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geqslant \delta_t/3\,, \tag{4.3.7}$$

and by Proposition 3.3.10 there exists $c_{1,t} > 0$ such that

$$Q_n(t) \leqslant \frac{2}{c_{1,t}} e^{nP_*(t)}, \ \forall n \geqslant 1, \tag{4.3.8}$$

Observe that (4.3.8) together with (4.2.8) give the upper bound

$$\mathcal{G}_n^\delta(W,t) \leqslant \frac{4}{C_1\delta} Q_n(t) \leqslant \frac{8}{C_1\delta c_{1,t}} e^{nP_*(t)}, \ \forall n \geqslant 1, \ \forall \delta \leqslant \delta_0. \tag{4.3.9}$$

Finally, (4.3.1) and (4.3.7) imply the following lower bound for any scale $\delta = \delta_0/2^N$.

**Lemma 4.3.5.** *For all $t \in (0, t_\infty)$ and $\delta = \delta_0/2^N$, there exists $c_{0,t}(\delta) > 0$ such that*

$$\mathcal{G}_n^\delta(W,t) \geqslant c_{0,t}(\delta) e^{nP_*(t)}, \ \forall n \geqslant 1, \ \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| \geqslant \delta/3. \tag{4.3.10}$$

*The time reversal of the statement holds for $T^{-1}$.*

*Proof.* First, assume $\delta < \delta_t$. Each element of $L_n^{\delta_t}(W)$ contains at least $\delta_t/(3\delta)$ elements of $\mathcal{G}_n^\delta(W)$. So if $|W| \geqslant \delta_t/3$, then (4.3.1) and bounded distortion for $\tau$ give

$$\mathcal{G}_n^\delta(W,t) \geqslant \frac{e^{-tC}\delta_t}{3\delta} L_n^{\delta_t}(W,t) \geqslant \frac{e^{-tC}\delta_t}{4\delta} \mathcal{G}_n^{\delta_t}(W,t) \geqslant \frac{e^{-tC}\delta_t c_{0,t}}{4\delta} e^{nP_*(t)}, \tag{4.3.11}$$

for all $n \geqslant n_t$, where we have used (4.3.7) in the last step.

Next, if $|W| \in [\delta/3, \delta_t/3)$, then there exists $n_W \leqslant C' \log(\delta_t/\delta)$ such that $T^{-n_W}(W)$ has a connected component $V$ of length at least $\delta_t/3$. This is because while $T^{-n}W$ remains short, the number of components of $T^{-n}W$ is at most $Kn+1$ by (4.2.2) while $|T^{-n}W| \geqslant C_1\Lambda^n|W|$ according to (4.2.1). Thus setting $\bar{n} = \max\{n_W, n_t\}$, we apply (4.3.11) to $V$ to estimate for $n \geqslant \bar{n}$.

$$\mathcal{G}_n^\delta(W,t) \geqslant \mathcal{G}_{n-\bar{n}}^\delta(V,t) e^{-\bar{n}\tau_{\max}} \geqslant e^{-\bar{n}(\tau_{\max}+P_*(t))} e^{-tC} \frac{\delta_t}{4\delta} c_{0,t} e^{nP_*(t)},$$

which proves (4.3.10) by definition of $\bar{n}$. If $n < \bar{n}$, then trivially

$$\mathcal{G}_n^\delta(W,t) \geqslant e^{-n\tau_{\max}} \geqslant e^{-\bar{n}(\tau_{\max}+P_*(t))} e^{nP_*(t)}.$$

Finally, if $\delta \geqslant \delta_t$, then since each element of $\mathcal{G}_n^\delta(W)$ contains at most $3\delta/\delta_t$ elements of $L_n^{\delta_t}(W)$ and $S_n^{\delta_t}(W) \subset S_n^\delta(W)$, we have

$$\mathcal{G}_n^{\delta_t}(W,t) = S_n^{\delta_t}(W,t) + L_n^{\delta_t}(W,t) \leqslant S_n^\delta(W,t) + \frac{3\delta}{\delta_t}\mathcal{G}_n^\delta(W,t) \leqslant \Big(1 + \frac{3\delta}{\delta_t}\Big)\mathcal{G}_n^\delta(W,t),$$

which gives the required lower bound on $\mathcal{G}_n^\delta(W,t)$, applying (4.3.7).

The time reversed statement of the lemma follows immediately using the reversibility of the billiard, as explained earlier. □

### 4.3.2 Key Lemmas

In view of Lemma 4.3.7 below, we adapt [BD22, Lemma 3.10]:

**Lemma 4.3.6** (Leapfrogging via the Hölder Inequality)**.** *For all*[15] $t \geqslant t_*$ *and* $\kappa > 0$ *there exists* $\omega_\kappa = \omega_\kappa(t_*, t) > 0$ *such that for all* $W \in \widehat{\mathcal{W}}^s$ *with* $|W| \geqslant \delta_{t_*}/3$,

$$\mathcal{G}_n^\delta(W,t) \geqslant \frac{\omega_\kappa(t_*,t)}{\delta} \cdot e^{n(P_*(t_*)-(|P'_-(t_*)|+\kappa)(t-t_*))}, \qquad (4.3.12)$$
$$\forall \delta = \frac{\delta_0}{2^N} \leqslant \delta_{t_*}, \ \forall n \geqslant n_{t_*}.$$

*In addition, for each* $\delta = \frac{\delta_0}{2^N} < \delta_0$ *there exists* $\omega^*_\kappa = \omega^*_\kappa(t_*, t, \delta) > 0$ *such that for all* $W \in \widehat{\mathcal{W}}^s$ *with* $|W| \geqslant \delta/3$,

$$\mathcal{G}_n^\delta(W,t) \geqslant \omega^*_\kappa(t_*,t,\delta) \cdot e^{n(P_*(t_*)-(|P'_-(t_*)|+\kappa)(t-t_*))}, \ \forall n \geqslant 1. \qquad (4.3.13)$$

*Finally, the time reversals of* (4.3.12) *and* (4.3.13) *also hold for the billiard map* $T^{-1}$*.*

The proof gives constants $\omega_\kappa(t_*, t)$ and $\omega^*_\kappa(t_*, t, \delta)$ which tend to zero as $t \to \infty$ (because the constant $\eta$ in the proof tends to zero as $t \to \infty$).

*Proof.* We start with (4.3.12) (for $t \geqslant t_*$). Recall from the proof of (4.3.11) that for $u \in (0, t_\infty)$ and $\delta < \delta_u$, if $|W| \geqslant \delta_u/3$ and $n \geqslant n_u$, then

$$\mathcal{G}_n^\delta(W,u) \geqslant e^{-uC} \frac{\delta_u}{4\delta} c_{0,u} e^{nP_*(u)}, \ \forall \delta < \delta_u, \qquad (4.3.14)$$

since each $V_i \in L_n^{\delta_u}(W)$ contains at least $\delta_u/3\delta$ elements of $\mathcal{G}_n^\delta(W)$.

Now, for $s \in (0, t_*)$, taking $\eta(s,t,t_*) \in (0,1]$ such that $\eta t + (1-\eta)s = t_*$, the Hölder inequality gives $\sum_i a_i^{t_*} \leqslant (\sum_i a_i^t)^\eta (\sum_i a_i^s)^{1-\eta}$ for any positive numbers $a_i$. It follows that for all $\delta \leqslant \delta_{t_*}$, each $W \in \widehat{\mathcal{W}}^s$ with $|W| \geqslant \delta_{t_*}/3$ and any $n \geqslant n_{t_*}$,

$$\begin{aligned}
\mathcal{G}_n^\delta(W,t) &\geqslant \frac{(\mathcal{G}_n^\delta(W,t_*))^{1/\eta}}{(\mathcal{G}_n^\delta(W,s))^{(1-\eta)/\eta}} \\
&\geqslant \left(e^{-t_*C} \frac{\delta_{t_*}}{4\delta} c_{0,t_*} e^{nP_*(t_*)}\right)^{1/\eta} \left(\frac{8}{C_1 \delta c_{1,s}} e^{nP_*(s)}\right)^{1-1/\eta} \\
&= \frac{1}{\delta}\left(e^{-t_*C} \frac{\delta_{t_*}}{4} c_{0,t_*}\right)^{1/\eta} \left(\frac{8}{C_1 c_{1,s}}\right)^{1-1/\eta} e^{n(P_*(t_*)-P_*(s))\frac{1-\eta}{\eta}} e^{nP_*(t_*)}, \qquad (4.3.15)
\end{aligned}$$

where we used (4.3.14) with $u = t_*$ for the lower bound in the numerator, and (4.3.9) for $s$ for the upper bound in the denominator, recalling that $\{s, t_*\} \subset (0, t_\infty)$ and $\delta_{t_*} \leqslant \delta_1 < \delta_0$.

Since $\eta(s,t,t_*) = (t_* - s)/(t - s)$, we have

$$(P_*(t_*) - P_*(s))\frac{1-\eta}{\eta} = \frac{t-t_*}{t_*-s}(P_*(t_*) - P_*(s)).$$

Fix $\kappa > 0$ and choose $s = s(\kappa, t_*) \in (0,1)$ close enough to $t_*$ (i.e. small enough $\eta_\kappa = \eta(s(\kappa,t_*), t, t_*) > 0$) such that (recalling $0 < s < t_*$ and $P'_-(u) < 0$ for all $u > 0$)

$$(P_*(s) - P_*(t_*))/(t_* - s) \leqslant |P'_-(t_*)| + \kappa. \qquad (4.3.16)$$

The bound (4.3.12) follows, setting, for $s = s(\kappa, t_*)$ (recall that $\eta_\kappa$ depends on $t$),

$$\omega_\kappa(t_*,t) = \left(e^{-t_*C} \frac{\delta_{t_*}}{4} c_{0,t_*}\right)^{1/\eta_\kappa} \left(\frac{8}{C_1 c_{1,s}}\right)^{1-1/\eta_\kappa}.$$

15. The same proof works replacing $t_*$ by an arbitrary number in $(0, t_\infty)$, as long as $t \geqslant t_*$.

For (4.3.13), we use that (4.3.9) for $s$ implies that for any $\delta \in (0, \delta_{t_*})$, for each $W \in \widehat{\mathcal{W}}^s$ with $|W| \geqslant \delta/3$, and all $n \geqslant 1$,

$$\mathcal{G}_n^\delta(W,t) \geqslant \frac{(\mathcal{G}_n^\delta(W,t_*))^{1/\eta}}{(\mathcal{G}_n^\delta(W,s))^{(1-\eta)/\eta}} \geqslant \left(c_{0,t_*}(\delta) \cdot e^{nP_*(t_*)}\right)^{1/\eta} \left(\frac{8}{C_1 \delta c_{1,s}} e^{nP_*(s)}\right)^{(\eta-1)/\eta}, \quad (4.3.17)$$

where we used (4.3.10) for $t_*$. We conclude by taking $s = s(\kappa, t_*) \in (0,1)$ close enough to $t_*$ such that (4.3.16) holds, setting (again, $\eta_\kappa$ depends on $t$)

$$\omega_\kappa^*(t_*, t, \delta) = c_{0,t_*}(\delta)^{1/\eta_\kappa} (8)^{1-1/\eta_\kappa} (C_1 \delta c_{1,s})^{1/\eta_\kappa - 1}\,.$$

□

Our second key lemma is inspired by [BD22, Lemma 3.11] (the proof below requires a more involved decomposition of orbits):

**Lemma 4.3.7.** *Let $t_* < t_\infty$ and $s_*(t_*) > t_\infty$ be as in Lemma 4.3.3. If $P(t_*) \geqslant 0$ then the SSP conditions (4.3.1), (4.3.2), and (4.3.3) hold at all $t \in [t_*, s_*)$ for $\varepsilon = 1/4$.*

*Proof of Lemma 4.3.7.* We first consider condition (4.3.1) of SSP.1.

By definition of $s_*$ (recall that $\inf |P'_-(s)| > -\log \theta_0/2$)

$$\theta_0^{t'/2} e^{|P'_-(t_*)|(t'-t_*)} < 1\,, \quad \forall t_* \leqslant t' < s_*\,. \quad (4.3.18)$$

Thus for all $t' \in [t_*, s_*)$ there exists $\kappa_1 = \kappa(t_*, t') > 0$ such that

$$\bar{\varepsilon} := \sup_{t_* \leqslant t \leqslant t'} \left(\theta_0^{t/2} e^{(|P'_-(t_*)| + \kappa_1)(t-t_*)}\right) < 1\,. \quad (4.3.19)$$

For $m_1 \geqslant \max\{n_0(t_*, \theta_0), n_{t_*}\}$ to be chosen later depending on $\varepsilon = 1/4$, $\bar{\varepsilon}$, $\delta_{t_*}$, and $\kappa_1$, pick $\delta_3(m_1) \in (0, \delta_{t_*}]$ (similarly to the choice of $\bar{\delta}_0$ in the proof of Lemma 4.2.1) so small that any stable curve of length at most $\delta_3$ can be cut into at most $Kj+1$ connected components by $\mathcal{S}_{-j}$ for $0 \leqslant j \leqslant 2m_1$.

For $n \geqslant m_1$, write $n = \ell m_1 + r$, for some $0 \leqslant r < m_1$ and $\ell \geqslant 1$. Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geqslant \delta_3/3$. We group the curves $W_i \in S_n^{\delta_3}(W)$ with $|W_i| < \delta_3/3$, as in the proof of [BD22, Lemma 3.11], according to the largest $k \in \{0, \dots, \ell-1\}$ such that $T^{(\ell-k)m_1+r} W_i \subset V_j \in L_{km_1}^{\delta_3}(W)$ (such a $k$ must exist since $|W| \geqslant \delta_3/3$ while $|W_i| < \delta_3/3$). Denote[16] by $\bar{\mathcal{I}}_{(\ell-k)m_1+r}^{\delta_3}(V_j)$ the set of $W_i \in \mathcal{G}_n^{\delta_3}(W)$ thus associated with $V_j \in L_{km_1}^{\delta_3}(W)$ (such elements are known to be small only at iterates $jm_1+r$). For such $W_i$, $T^{(\ell-k')m_1+r}(W_i)$ is contained in an element of $\mathcal{G}_{m_1k'}^{\delta_3}(W)$ shorter than $\delta_3/3$ for $k' < k$. So for $k > 0$, we may apply the inductive bound (4.2.6) since elements of $\bar{\mathcal{I}}_{(\ell-k)m_1+r}^{\delta_3}(V_j)$ can only be created by intersections with $\mathcal{S}_{-m_1}$ at the first $\ell - k - 1$ iterates and with $\mathcal{S}_{-m_1-r}$ at the last step. For $k = 0$, $W$ itself may be longer than $\delta_3$. Thus we first subdivide $W$ into at most $\delta_0/\delta_3$ curves of length at most $\delta_3$ and then apply (4.2.6) to each piece. This yields, for $t_* \leqslant t \leqslant t'$,

$$\begin{aligned} S_n^{\delta_3}(W,t) &\leqslant \sum_{k=0}^{\ell-1} \sum_{V_j \in L_{km_1}^{\delta_3}(W)} |e^{-tS_{km_1}\tau}|_{C^0(V_j)} \sum_{W_i \in \bar{\mathcal{I}}_{(\ell-k)m_1+r}^{\delta_3}(V_j)} |e^{-tS_{(\ell-k)m_1+r}\tau}|_{C^0(W_i)} \\ &\leqslant \frac{\delta_0}{\delta_3} \theta_0^{tn} + \sum_{k=1}^{\ell-1} \sum_{V_j \in L_{km_1}^{\delta_3}(W)} |e^{-tS_{km_1}\tau}|_{C^0(V_j)} \theta_0^{t((\ell-k)m_1+r)}\,. \end{aligned} \quad (4.3.20)$$

16. Note that $\bar{\mathcal{I}}_{(\ell-k)m_1+r}^{\delta}(V_j)$ was abusively denoted $\mathcal{I}_{(\ell-k)m_1+r}^{\delta}(V_j)$ in the proof of [BD20, Lemma 5.2], see footnote 23 there.

Next, recalling (4.2.3), for any $k \geqslant 1$, each $V_j \in L^{\delta_3}_{km_1}(W)$ is contained in an element $U_i \in \mathcal{G}^{\delta_{t*}}_{km_1}(W)$. Since $|V_j| \geqslant \delta_3/3$, there are at most $3\delta_{t_*}/\delta_3$ different $V_j$ corresponding to each fixed $U_i$. Then we group each $U_i \in \mathcal{G}^{\delta_{t*}}_{km_1}(W)$ according to its most recent long ancestor $W_a \in L^{\delta_{t*}}_j(W)$ for some $j \in [0, km_1]$. Note that $j = 0$ is possible if $|W| \geqslant \delta_{t_*}/3$. If $|W| < \delta_{t_*}/3$, and no such time $j$ exists for $U_i$, then by convention we also associate the index $j = 0$ to such $U_i$. In either case, $U_i \in \mathcal{I}^{\delta_{t*}}_{km_1}(W)$, and we may apply (4.2.6) after possibly subdividing $W$ into at most $\delta_0/\delta_{t_*}$ curves of length at most $\delta_{t_*}$. Then, for $j \geqslant 1$, we apply (4.2.7) from Lemma 4.2.2 to each $\mathcal{I}^{\delta_{t*}}_{km_1-j}(\cdot)$ (since $\delta_3 \leqslant \delta_{t_*}$, the constant $m_1(\delta_{t_*}) \leqslant m_1(\delta_3)$, so the bound holds with our chosen $m_1$, although it may not be optimal),

$$\begin{aligned} L^{\delta_3}_{km_1}(W,t) &\leqslant \frac{3\delta_{t_*}}{\delta_3}\Bigg( \sum_{U_i \in \mathcal{I}^{\delta_{t*}}_{km_1}(W)} |e^{-tS_{km_1}\tau}|_{C^0(U_i)} \\ &\qquad + \sum_{j=1}^{km_1} \sum_{W_a \in L^{\delta_{t*}}_j(W)} |e^{-tS_j\tau}|_{C^0(W_a)} \sum_{U_i \in \mathcal{I}^{\delta_{t*}}_{km_1-j}(W_a)} |e^{-tS_{km_1-j}\tau}|_{C^0(U_i)} \Bigg) \\ &\leqslant \frac{3\delta_{t_*}}{\delta_3}\left( \frac{\delta_0}{\delta_{t_*}} \theta_0^{tkm_1} + \sum_{j=1}^{km_1} \sum_{W_a \in L^{\delta_{t*}}_j(W)} |e^{-tS_j\tau}|_{C^0(W_a)} K m_1 \theta_0^{t(km_1-j)} \right). \end{aligned}$$

Combining this estimate with (4.3.20) yields (summing over $k$ for the $j = 0$ terms and adding the term corresponding to $k = 0$),

$$S^{\delta_3}_n(W,t) \leqslant \frac{3\delta_0}{\delta_3}\frac{n}{m_1}\theta_0^{tn} + \frac{3\delta_{t_*}}{\delta_3}\sum_{k=1}^{\ell-1}\sum_{j=1}^{km_1} K m_1 \theta_0^{t(n-j)} L^{\delta_{t*}}_j(W,t). \tag{4.3.21}$$

For fixed $k \in \{1, \dots, \ell-1\}$, and for each $1 \leqslant j \leqslant km_1$ such that $L^{\delta_{t*}}_j(W) \neq \emptyset$, the lower bound (4.3.12) in Lemma 4.3.6 and the distortion constant $e^{-tC} \geqslant e^{-t'C}$ imply (note that $n - j \geqslant \ell m_1 + r - km_1 \geqslant r + m_1 \geqslant n_{t_*}$),

$$\begin{aligned} \mathcal{G}^{\delta_3}_n(W,t) &\geqslant \sum_{W_a \in L^{\delta_{t*}}_j(W)} e^{-tC}|e^{-tS_j\tau}|_{C^0(W_a)} \sum_{W_i \in \mathcal{G}^{\delta_3}_{n-j}(W_a)} |e^{-tS_{n-j}\tau}|_{C^0(W_i)} \\ &\geqslant \frac{\omega_{\kappa_1}(t_*,t)}{\delta_3 e^{t'C}} e^{(n-j)(P_*(t_*)-(|P'_-(t_*)|+\kappa_1)(t-t_*))} \sum_{W_a \in L^{\delta_{t*}}_j(W)} |e^{-tS_j\tau}|_{C^0(W_a)}. \end{aligned} \tag{4.3.22}$$

Combining (4.3.21) with either (4.3.22) (for $j \geqslant 1$) or (4.3.13) from Lemma 4.3.6 (for

$j = 0$) and setting $\Delta = 3e^{t'C}\delta_{t_*}Km_1$, yields (using that $P(t_*) \geqslant 0$),

$$\begin{aligned}\frac{S_n^{\delta_3}(W,t)}{\mathcal{G}_n^{\delta_3}(W,t)} &\leqslant n\frac{\frac{3\delta_0}{\delta_3 m_1}\theta_0^{tn}}{\omega^*_{\kappa_1}(t_*,t,\delta_3)e^{n(P_*(t_*)-(|P'_-(t_*)|+\kappa_1)(t-t_*))}} \\ &\quad + \sum_{k=1}^{\ell-1}\sum_{j=1}^{km_1}\frac{\frac{3\delta_{t_*}}{\delta_3}Km_1\theta_0^{t(n-j)}L_j^{\delta_{t_*}}(W,t)}{\frac{\omega_{\kappa_1}(t_*,t)}{\delta_3 e^{t'C}}e^{(n-j)(P_*(t_*)-(|P'_-(t_*)|+\kappa_1)(t-t_*))}L_j^{\delta_{t_*}}(W,t)} \\ &\leq \frac{3\delta_0}{\delta_3\cdot\omega^*_{\kappa_1}(t_*,t,\delta_3)\cdot m_1}n(e^{-P_*(t_*)}\bar\varepsilon)^n + \frac{\Delta}{\omega_{\kappa_1}(t_*,t)}\sum_{k=1}^{\ell-1}\sum_{j=1}^{km_1}(e^{-(P_*(t_*)}\bar\varepsilon)^{n-j} \\ &\leqslant \frac{3\delta_0}{\delta_3\cdot\omega^*_{\kappa_1}(t_*,t,\delta_3)\cdot m_1}n\bar\varepsilon^n + \frac{\Delta}{\omega_{\kappa_1}(t_*,t)}\frac{1}{1-\bar\varepsilon}\sum_{k=1}^{\ell-1}\bar\varepsilon^{n-km_1} \\ &\leqslant \frac{3\delta_0}{\delta_3\cdot\omega^*_{\kappa_1}(t_*,t,\delta_3)\cdot m_1}n\bar\varepsilon^n + \frac{3e^{t'C}\delta_{t_*}Km_1}{\omega_{\kappa_1}(t_*,t)\cdot}\frac{\bar\varepsilon^{m_1}}{(1-\bar\varepsilon)(1-\bar\varepsilon^{m_1})}\,. \end{aligned} \tag{4.3.23}$$

To establish (4.3.1), choose first $m_1 \geqslant n_{t_*}$ such that the second term is less than $\frac{\varepsilon}{2}$, setting $\delta_t := \delta_3(m_1)$, and then $n_t \geqslant m_1$ such that the first term is less than $\frac{\varepsilon}{2}$ for $n \geqslant n_t$.

We next show (4.3.2). For $n \geqslant n_t$, we deduce from (4.3.1) and (4.3.13) (for small $\kappa > 0$) that, for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geqslant \delta_t/3$,

$$L_n^{\delta_t}(W,t) \geqslant \frac{3}{4}\mathcal{G}_n^{\delta_t}(W,t) \geqslant \frac{3}{4}\omega^*_\kappa(t_*,t,\delta_t)e^{nP_*(t_*)}e^{-n(t-t_*)(|P'_-(t_*)|+\kappa)}\,.$$

Since $e^{-|P'_-(t_*)|(t-t_*)} > \theta_0^{t/2} \geqslant e^{-t\tau_{\min}/2}$ by (4.3.18), while $P_*(t_*) \geqslant 0$, it suffices to take $\kappa$ such that $(t-t_*)\kappa + \frac{t}{2}\tau_{\min} < t\tau_{\min}$ to complete the proof of (4.3.2).

It remains to consider SSP.2. We may assume $|W| < \delta_{t_*}/3$ since otherwise (4.3.1) from SSP.1 implies (4.3.3) with $n_t^* = n_t$. As observed in the proof of [BD20, Cor. 5.3], there exists $\bar C_2$ (depending only on the billiard table) such that the first iterate $\ell = \ell_0$ at which $\mathcal{G}_\ell^{\delta_{t_*}}(W)$ contains at least one element of length more than $\delta_{t_*}/3$ satisfies

$$\ell_0 \leqslant n_2 = n_2(\delta_{t_*}) := \bar C_2|\log(|W|/\delta_{t_*})|\,.$$

Since $|W| < \delta_{t_*}/3$, it suffices to consider the term corresponding to $j = 0$ (and $k = 0$) in (4.3.23) (the other one is bounded by $\varepsilon/2$ for $n \geqslant m_1$ for $m_1$ chosen as above). For this purpose, for any $n = \ell m_1 + r \geqslant m_1$, the first term of (4.3.21) is replaced by

$$\frac{\delta_{t_*}}{3\delta_3}\theta_0^{tn} + \sum_{k=1}^{\ell-1}\frac{3\delta_{t_*}}{\delta_3}\theta_0^{tn} \leqslant \frac{3\delta_{t_*}n}{\delta_3 m_1}\theta_0^{tn}\,, \tag{4.3.24}$$

where we have applied (4.2.6) from Lemma 4.2.2. For any $n \geqslant \max\{n_2, m_1\}$, the bound (4.3.13) from Lemma 4.3.6 is replaced by

$$\mathcal{G}_n^{\delta_3}(W,t) \geqslant \omega^*_{\kappa_1}(t_*,t,\delta_3)\cdot e^{-tn_2\tau_{\max}}e^{(n-n_2)(P_*(t_*)-(|P'_-(t_*)|+\kappa_1)(t-t_*))}\,. \tag{4.3.25}$$

Dividing (4.3.24) by (4.3.25), the term corresponding to $j = 0$ in (4.3.23) is bounded by

$$\begin{aligned}&\frac{3\delta_{t_*}\frac{n}{m_1}\theta_0^{tn}}{\delta_3\cdot\omega^*_{\kappa_1}(t_*,t,\delta_3)\cdot e^{-tn_2\tau_{\max}}e^{(n-n_2)(P_*(t_*)-(|P'_-(t_*)|+\kappa_1)(t-t_*))}} \\ &\qquad\leqslant \frac{3\delta_{t_*}e^{tn_2\tau_{\max}}}{m_1\cdot\omega^*_{\kappa_1}(t_*,t,\delta_3)\cdot\delta_3}n\bar\varepsilon^{n-n_2}\,.\end{aligned}$$

We conclude, since, if $n_t^*/n_2$ is large enough (depending on $t$, $\bar{\varepsilon}$, $\delta_3 = \delta_t$) then

$$n(\bar{\varepsilon}^{n/n_2} e^{t\tau_{\max}})^{n_2} < \frac{\varepsilon}{2} \cdot \frac{\bar{\varepsilon}^{n_2} \cdot m_1 \cdot \delta_3 \cdot \omega_{\kappa_1}^*(t_*, t, \delta_3)}{3\delta_{t_*}}\,, \ \forall n \geqslant n_t^*\,.$$

□

### 4.3.3 Theorem 4.1.4: Proof of Lemma 4.3.2

In view of the discussion above Lemma 4.3.2, it only remains to show Lemma 4.3.2 to establish Theorem 4.1.4:

*Proof of Lemma 4.3.2.* If $P(t_\infty) < 0$ we are done, as explained before Lemma 4.3.3. Assume for a contradiction that $P(t_\infty) \geqslant 0$. Let $t_* < t_\infty$ and $s_*(t_*) > t_\infty$ be as in Lemma 4.3.3, and fix $t_\infty < t_2 < s_*$. Then Lemma 4.3.7 applied to $\varepsilon = 1/4$ gives that the SSP conditions (4.3.1), (4.3.2), and (4.3.3) hold for all $t \in [0, t_2]$. Since $t_2 > t_\infty$, this is a contradiction. □

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